மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

MANONMANIAM SUNDARANAR UNIVERSITY

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தொலைநிலை தொடர்கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND

CONTINUING EDUCATION

M.Sc. MATHEMATICS

I YEAR

REAL ANALYSIS-I

Sub. Code: SMAM12 **(For Private Circulation only)**

M.SC. MATHEMATICS - I YEAR

SMAM12: REAL ANALYSIS-I

SYLLABUS

Unit I

Functions of bounded variation: Introduction - Properties of monotonic functions -Functions of bounded variation - Total variation - Additive property of total variation – Total variation on [a, x] as a function of x - Functions of bounded variation expressed as the difference of two increasing functions - Continuous functions of bounded variation.

Chapter 6: Sections 6.1 to 6.8

Infinite Series: Absolute and conditional convergence – Dirichlet's test and Abel's test - Rearrangement of series – Riemann's Theorem on conditionally convergent series.

Chapter 8: Sections 8.8, 8.15, 8.17, 8.18

Unit II

The Riemann - Stieltjes Integral: Introduction - Notation - The definition of the Riemann - Stieltjes integral - Linear Properties - Integration by parts- Change of variable in a Riemann - Stieltjes integral - Reduction to a Riemann Integral – Euler's summation formula - Monotonically increasing integrators, Upper and lower integrals - Additive and linearity properties of upper, lower integrals – Riemann's condition - Comparison theorems.

Chapter – 7: Sections 7.1 to 7.6, 7.11-7.14

Unit III

The Riemann-Stieltjes Integral - Integrators of bounded variation-Sufficient conditions for the existence of Riemann-Stieltjes Integrals-Necessary conditions for the existence of RS integrals- Mean value theorems -integrals as a function of the interval –Second fundamental Theorem of integral calculus-Change of variable -Second Mean Value Theorem for Riemann integral- Riemann-Stieltjes integrals depending on a parameter.

Chapter - 7: Sections 7.15 - 7.23

Unit IV

Infinite Series and infinite Products - Double sequences - Double series -Rearrangement Theorem for double series - A sufficient condition for equality of iterated series - Multiplication of series – Cesaro summability - Infinite products.

Chapter 8: Sections 8.20, 8.21 to 8.26

Power series - Multiplication of power series - The Taylor's series generated by a function - Bernstein's Theorem

Chapter 9: Sections 9.14, 9.15, 9.19, 9.20

Unit V

Sequences of Functions – Pointwise convergence of sequences of functions -Examples of sequences of real - valued functions - Uniform convergence and continuity -Cauchy condition for uniform convergence - Uniform convergence of infinite series of functions - Riemann - Stieltjes integration – Non-uniform Convergence and Term-by-term Integration - Uniform convergence and differentiation - Sufficient condition for uniform convergence of a series - Mean convergence.

Chapter -9: Sections 9.1 to 9.6, 9.9, 9.10, 9.11.

Text Book

Tom M. Apostol: Mathematical Analysis, 2nd Edition, Addison-Wesley Publishing Company Inc. New York, 1974.

M.Sc. MATHEMATICS –I YEAR

SMAM12: REAL ANALYSIS-I

SYLLABUS

Unit I

Unit II

Unit V

Unit I

Functions of bounded variation: Introduction - Properties of monotonic functions -Functions of bounded variation - Total variation - Additive property of total variation – Total variation on [a, x] as a function of x - Functions of bounded variation expressed as the difference of two increasing functions - Continuous functions of bounded variation.

Infinite Series: Absolute and conditional convergence – Dirichlet's test and Abel's test - Rearrangement of series – Riemann's Theorem on conditionally convergent series.

Functions of Bounded Variation

1.1. Introduction:

Let f be a real-valued function defined on a subset S of R. Then f is said to be increasing (or non-decreasing) on S if for every pair of points x and y in S,

$$x < y \Rightarrow f(x) \leq f(y)$$

If $x < y \Rightarrow f(x) < f(y)$, then f is said to be strictly increasing on S. (Decreasing functions are similarly defined.) A function is called monotonic on S if it is increasing on S or decreasing on S.

If f is an increasing function, then - f is a decreasing function. Because of this simple fact, in many situations involving monotonic functions it suffices to consider only the case of increasing functions.

Properties of Monotonic Functions

Theorem 1.2:

Let f be an increasing function defined on [a, b] and let $x_0, x_1 \ldots\ldots x_n$ be

n+1 points such that $a = x_0<x_1<x_2<\ldots\ldots<x_n=b$ Then we have the inequality.

$\sum_{k=1}^{n-1}[\ [f\ (x_k+) - f\ (x_k-)] \leq f(b) - f(a)$.

Proof:

Let ' f ' be an increasing function defined on [a, b]

Let $x_0, x_1, x_2 \ldots\ldots x_n$ be n+1 points

$a = x_0 < x_1 < x_2 < \ldots . < x_n = b$

To prove that: $\sum_{k=1}^{n-1} [\,[f(x_k+) - f(x_k-)] \leq f(b) - f(a)$

Let $y_1, y_2, \ldots . y_n \in [a, b]$ be points

such that $a = x_0 < y_1 < x_1 < y_2 < \ldots < y_k < x_k < y_{k+1} < \ldots . < y_n < x_n = b$

(i.e.) for each $K \in \{1,2,\ldots . n-1\}$,

we have, $y_k < x_k < y_{k+1}$

Let $f(x_k-) = \lim_{x \to x_k-} f(x)$ & $f(x_k^+) = \lim_{x \to x_k+} f(x)$

Since f is an increasing function,

$f(y_k) \leq f(x_k-) \leq f(x_k) \leq f(x_k+) \leq f(y_{k+1})$

$\therefore f(y_{k+1}) - f(y_k) \geq f(x_k+) - f(x_k-)$

(i.e.) $f(x_k+) = f(x_k-) \leq f(y_{k+1}) - f(y_k)$

$\sum_{k=1}^{n-1} [f(x_k+) - f(x_k-)] \leq \sum_{k=1}^{n-1} [f(y_{k+1}) - f(y_k)]$

$= [f(y_2) - f(y_1)] + [f(y_3) - f(y_2)] + \ldots . + [f(y_n) - f(y_{n-1})]$

$= [f(y_n) - f(y_{n-1})]$

$\therefore \sum_{k=1}^{n-1} [\,[f(x_k+) - f(x_k-)] \leq f(y_n) - f(y_1) \leq f(b) - f(a)$

[$\because$ f is increasing. $a < y \Rightarrow f(a) < f(y_1)$; $y_n < b \Rightarrow f(y_n) < f(b)$

Hence $\sum_{k=1}^{n-1} [f(x_k+) - f(x_k-)] \leq f(b) - f(a)$

Theorem 1.3:

If f is monotonic [a, b], then the set of discontinuities of *f* is countable.

Proof:

[There are two cases to consider - both of which are analogous so we will only consider the case when f is monotonically increasing]

Let f *be* an increasing function on [a, b]

We note that a discontinuity occurs at $x = [a, b]$

when $f(x-) \neq f(x+)$

In particular, since 'f' is an increasing function,

0 discontinuity occurs when $f(x+) - f(x-) > 0$

There exists a natural number m>o such that $0 < 1/m < f(x+)-f(x-)$

Let $S_m = \{ x \in (a ,b) : f(x+) - f[x-) > 1/m + m \in N^+ \}$

Let $x_1, x_2, ..., x_{n-1} \in S_m$ such that $x_1 < x_2 < .... < x_{n-1}$

Then $x_1 , x_2, ..., x_{n-1}$ are discontinuities of f such that their jump

$f(x_k+) - f(x_k-) > 1/m$

$\Rightarrow \sum_{k=1}^{n-1} 1/m \leq \sum_{k=1}^{n-1} f(x_k+) = f(x_k-)$

$\Rightarrow \frac{n-1}{m} \leq f(b)-f(a)$ (by theorem. 1.2)

$\Rightarrow n-1 \leq m[f(b)-f(a)]$

$\Rightarrow n \leq m [f(b) - f(a)] + 1$

$\therefore$The number of discontinuities `n' in S_m is bounded above.

$\therefore$ S_m must be a finite set of discontinuities.

(i.e.), S_m is a countable set of discontinuities.

$\therefore$ For all discontinuities $x \in (a, b)$, *there exist* $m \in N^+$ such that

$f(x+) - f(x-) > y_m$, the set of all discontinuities of f *on* (a, b) *is* $\bigcup_{m=1}^{\infty} S_m$

Each S_m is a countable set

$\therefore$ $\bigcup_{m=1}^{\infty} S_m$ is countable

Hence f has at most a countably infinite number of discontinuities.

Functions of Bounded Variation

Definition 1.4:

If $[a, b]$ is a compact interval, a set of points $P = \{x_0, x_1, x_2, ..., x_n\}$, satisfying the inequalities $a = x_0 < x_1 < ..... < x_{n-1} < x_n = b$, is called a partition of [a, b]. The interval $[x_{k-1}, x_k]$ is called the k^{th} sub interval of P and we write $\Delta x_k = x_k - x_{k-1}$

$\sum_{k=1}^{n} \Delta x_k = \sum_{k=1}^{n} x_k - x_{k-1} = (x_1 - x_0)(x_2 - x_1) + ... + (x_n - x_{n-1}) = x_n - x_0 = b - a$

$\sum_{k=1}^{n} \Delta x_k = b-a$. The collection of all partitions of [a, b] will be denoted by

$\mathcal{P}\ [a, b]$

Definition 1.5:

Let f be defined on $[a, b]$. If $P = \{x_0, x_1, x_2, ..., x_n\}$ is a partition of [a, b],

write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, ..., n$

If there exist a positive number M $\in \sum_{k=1}^{n} |\Delta f_k| \leq$ M

for all partitions of [a, b], then f is said to be of bounded variation on [a, b].

Theorem 1.6:

If f is monotonic on [a, b], then f is of bounded variation on [a, b].

Proof:

Let f be an increasing function on $[a, b]$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$

Then $\Delta f_k = f(x_k) - f(x_{k-1})$, for all $k = 1, 2, ..., n$

Now

$$\sum_{k=1}^{n} |\Delta f_k| \leq \; = \sum_{k=1}^{n} f_k \quad (\because \text{f is increasing})$$

$$= \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]$$

$$= [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \cdots + [f(x_n) - f(x_{n-1})]$$

$$= f(x_n) - f(x_0) < f(b) - f(a)$$

Let M= f(b) -f(a) > 0

$\sum_{k=1}^{n}|\Delta f_k| \leq M$, *M>0*

Hence *f* is of bounded variation on [a, b]

Theorem 1.7:

If *f* is continuous on [a, b] and if f' exists and is bounded in the interior, say

$|f'(x)| \leq A$ for all x in (a, b), then *f* is of bounded variation on [a, b].

Proof:

Let *f* be continuous on [a, b] and f' exists and bounded in (a, b)

(i.e.) $|f'(x)| \leq A$ *for all* $x \in (a, b)$.

Let p = $\{a=x_0, x_1, x_2, \ldots, x_n=b\}$ *be a partition of [a, b]*

Then $\Delta x_k = x_k - x_{k-1}$ & $\Delta f_k = f(x_k) - f(x_{k-1})$ Since 'f' is continuous on [a, b] & f' exists in (a,b) & by mean value theorem,

$f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1})$ *for all* $t_k \in (x_{k-1}, x_k)$ ....(1)

T0 prove: *f* is of bounded variation on [a, b]

Now, $\sum_{k=1}^{n}|\Delta f_k|$ $=\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})|$

$=\sum_{k=1}^{n}|\Delta f'_k|.|x_k - x_{k-1}|$

$=A.\sum_{k=1}^{n}(x_k - x_{k-1})$

$=A.\{(x_1-x_0] + [x_2-x_1] + \ldots\ldots + [x_n-x_{n-1}]\}$

$=A.(x_n-x_0)$

$=A.(b-a)$

$\sum_{k=1}^{n}|\Delta f_k| \leq A.(b-a)$

Let $M = A(b-a) > o$

$\therefore \sum_{k=1}^{n}|\Delta f_k| \leq M$, *M>0* Hence *f* is of bounded variation on [a,b]

Theorem 1.8:

If f is of bounded variation on [a, b], say $\sum_{k=1}^{n} |\Delta f_k| \leq M$ for all partitions of [a, b], then f is bounded on [a, b] In fact, $|f(x) \leq |f(a)| + M$ for all x in [a, b]

Proof:

Let f be of bounded variation on [a, b]

(ie.) $\sum_{k=1}^{n} |\Delta f_k| \leq M$ for all partitions of [a, b]

$\Rightarrow \sum_{k=1}^{n} | f(x_k) - f(x_{k-1})| \leq M$ ....(1)

Consider the partition P= {a,x,b} for all $x \in$ [a, b]

By (1), we get,

$|[f(x) - f(a)]| + |[f(b) - f(x)]| \leq M$

$\Rightarrow |f(x) - f(a)| \leq M$ $\quad (\because |f(x)-f(a)| \leq |f(x)+(a)| + |f(b)-f(x)|)$

$\Rightarrow ||f(x)| - |f(a)|| \leq M$ $\quad (\because |x-y| \geq ||x|-|y||)$

$\Rightarrow |f(x)| - |f(a)| \leq M$ $\quad (\because x \leq |x|)$

$\Rightarrow |f(x)| \leq |f(a)| + M$ *for all* $x \in [a, b]$

Examples 1.9:

1.Construct a continuous function which is not of bounded variation

Consider the function $f(x) = \begin{cases} x \cos(\pi/2x) & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

Here f is continuous on [0,1]

Consider the partition into 2n sub intervals

$P = \{0, \frac{1}{2n}, \frac{1}{2n}-1 \ldots\ldots, 1/3, 1/2, 1\}$

We know that

$|\cos(\pi/2x)| = \begin{cases} 0\ if\ x = \frac{1}{2k-1} \\ 1\ if\ x = \frac{1}{2k} \end{cases}$ *for all* $k = 1,2,\ldots\ldots n$

Now,

$|\Delta f_1|=|f(\frac{1}{2n})-f(0)| = |f(1/2n)- 0| = 1/2n$

$|\Delta f_2|=|f(\frac{1}{2n}-1)-f(\frac{1}{2n})| = |0- \frac{1}{2n}| = 1/2n$

$|\Delta f_3|=|f(\frac{1}{2n}-2)-f(\frac{1}{2n}-1)| = |f(\frac{1}{2n}-2)- 0| = 1/2n-2$

$|\Delta f_4|=|f(\frac{1}{2n}-3)-f(\frac{1}{2n}-2)| = |0 - f(\frac{1}{2n}-2)| = 1/2n-2$

….

….

$|\Delta f_{2n-1}|=|f(\frac{1}{2})-f(\frac{1}{3})| = |f(1/2)- 0| = \frac{1}{2}$

$|\Delta f_{2n}|=|f(1)-f(\frac{1}{2})| = |0 - f(\frac{1}{2})| = \frac{1}{2}$

$\therefore \sum_{k=1}^{n} |\Delta f_k| = \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \dots\dots\dots\dots + \frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots\dots + \frac{1}{n-1} + \frac{1}{n}$

(i.e.) $\sum_{k=1}^{n} |\Delta f_k| = 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots\dots + \frac{1}{n-1} + \frac{1}{n}$

This is not bounded for all 'n' [$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges]

In this example, f' exists in (0,1) but f' is not bounded on (0,1)

Hence f' is not of bounded variation on [0,1]

However, f' bounded on any compact interval not containing the origin and hence f will be of bounded variation on such an interval

2. Construct a continuous function which is of bounded variation**.**

Consider the function f(x) = $\begin{cases} x^2 \cos(1/x) & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

Here f is continuous on [0,1]

Also, $f'(0) = 0$

For $x \neq 0$, $f'(x) = x^2 + (-\sin(\frac{1}{x}).-1/x^2 + \cos(\frac{1}{x}).2x$

$f'(x) = \sin\frac{1}{x} + 2x\cos\frac{1}{x}$

$|f'(x)| = |\sin\frac{1}{x} + 2\cos\frac{1}{x}|$

$\leq |\sin\frac{1}{x}| + 2|x|.|\cos\frac{1}{x}|$

$\leq 1 + 2.1.1 = 3$

$|f'(x)| \leq 3$

f' exists and is bounded in [0,1]

Hence *f* is of bounded variation on [0,1]

3. Boundedness of *f* is not necessary for *f* to be of bounded variation

Consider a function $f(x) = x^{1/3}$

Let x<y

Now, x<y

$x^{1/3} < y^{1/3}$

$f(x) \leq f(y)$

f is a monotonic increasing function

Hence *f* is of bounded variation on every finite interval

However $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \rightarrow \infty$ *as* $x \rightarrow 0$

Total Variation

Definition 1.10:

Let *f* be of bounded variation on [a, b] and let $\sum(P)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P= $\{x_0, x_1, x_2, \ldots, x_n\}$ of [a ,b]. Then the number

$V_f(a, b) = sup(\sum\{P): P \in \mathcal{P}[a,b]\}$, is called the total variation of *f* on the interval [a,b].

Note 1.11:

* We write V_f instead of V_f (a, b)

* Since *f* is of bounded variation on [a, b], then the total variation V_f is finite.

*Since each sum $\sum(P) \geq 0$, $V_f \geq 0$

* V_f (a, b) =0 ⇔ f is constant on [a, b]

Theorem1.12:

Assume that *f* and *g* are each of bounded variation on [a, b]. Then so are their sum, difference, and product. Also we have where $V_{f\pm g}$ & $V_f + V_g$ &

$V_{f.g} \leq AV_F + BV_g$, A = sup {|g(x)|:x∈[a,b]}, B = sup {|f(x)|:x∈[a,b]}

Proof:

Given that f and g are each of bounded variation on [a, b]

⇒there exist a positive numbers $M_1, M_2 > 0$ such that for all partitions

P= $\{a=x_0, x_1, x_2, \ldots, x_n= b\}$, we have

$\sum_{k=1}^{n} |\Delta f_k| \leq M_1$, & $\sum_{k=1}^{n} |\Delta f_k| < M_2$

(i.e.) $\sum_{k=1}^{n} | f(x_k)- f(x_{k-1})| \leq M_1$, $\sum_{k=1}^{n} |g(x_k- g(x_{k-1})| \leq M_2$

(i)To prove: f+g is of bounded variation & $V_{f+g} \leq V_f + V_g$

Let h=f+g

Now,

$$\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |h(x_k)- h(x_{k-1})|$$

$$= \sum_{k=1}^{n} |(f+g)(x_k) - (f+g)(x_{k-1})|$$

$$= \sum_{k=1}^{n} | f(x_k)+ g(x_k) - (f(x_{k-1})+ g(x_{k-1})|$$

$$= \sum_{k=1}^{n} |[f(x_k)- f(x_{k-1})]+[g(x_k)-g(x_{k-1})]|$$

$$\leq \sum_{k=1}^{n}\{|f(x_k)- f(x_{k-1})|+|g(x_k)-g(x_{k-1})|\}$$

$$\leq \sum_{k=1}^{n} | f(x_k)- f(x_{k-1})]|+\sum_{k=1}^{n} |g(x_k)-g(x_{k-1})| \quad \ldots\ldots\ldots\ldots(2)$$

$\leq$ M_1+M_2

$\therefore \sum_{k=1}^{n} |\Delta h_k| \leq$ M_1+M_2

Let $M=M_1+M_2 >0$

Hence h=f+g is of bounded variation on [a, b]

From equation (1)

$\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k|$

Taking supremum, we get

$\Rightarrow V_h \leq V_f + V_g$

(i.e.) $V_{f+g} \leq V_f + V_g$

(ii) To prove: f-g is of bounded variation & $V_{f-g} \leq V_f + V_g$

Let h=f-g

Now, $\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |h(x_k)- h(x_{k-1})|$

$\leq \sum_{k=1}^{n} |(f-g)(x_k)-(f-g)(x_{k-1})|$

$= \sum_{k=1}^{n} | f(x_k)- g(x_k) - (f(x_{k-1})- g(x_{k-1})|$

$= \sum_{k=1}^{n} |[f(x_k)- f(x_{k-1})]+[g(x_{k-1})-g(x_k]|$

$\leq \sum_{k=1}^{n} | f(x_k)- f(x_{k-1})|+|g(x_k)-g(x_{k-1})|$

$\leq \sum_{k=1}^{n} | f(x_k)-f(x_{k-1})]|+\sum_{k=1}^{n} |g(x_k)-g(x_{k-1})|$ ………(2)

$\leq M_1+M_2$

$\therefore \sum_{k=1}^{n} |\Delta h_k| \leq$ M_1+M_2

Let $M=M_1+M_2 >0$

Hence h=f-g is of bounded variation on [a, b]

From equation (2)

$\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} | f(x_k)- f(x_{k-1})]|+\sum_{k=1}^{n} |g(x_k)-g(x_{k-1})|$

` $\leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k|$

Taking supremum, we get,

$\Rightarrow V_h \leq V_f + V_g$

(i.e.) $V_{f\text{-}g} \leq V_f + V_g$

(iii)fg is of bounded variation & $V_{f.g} \leq A\, V_f + B\, V_g$

Let h=fg

Now

$\sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})|$

$= \sum_{k=1}^{n} |(fg)(x_k) - (fg)(x_{k-1})|$

$= \sum_{k=1}^{n} | f(x_k).g(x_k) - f(x_{k-1}).\, g(x_k) + f(x_{k-1}).\, g(x_k) - f(x_{k-1}).\, g(x_{k-1})|$

$= \sum_{k=1}^{n} |[f(x_k) - f(x_{k-1})]g(x_k) + [g(x_{k-1}) - g(x_k)]f(x_{k-1})|$

$= \sum_{k=1}^{n} |[f(x_k) - f(x_{k-1})]g(x_k)| + |[g(x_{k-1}) - g(x_k]f(x_{k-1})|$

$\leq \sum_{k=1}^{n} |[f(x_k) - f(x_{k-1})]g(x_k)| + \sum_{k=1}^{n} |[g(x_{k-1}) - g(x_k]f(x_{k-1})|$

……… (3)

$\leq AM_1 + BM_2$

$\therefore \sum_{k=1}^{n} |\Delta h_k| \leq AM_1 + BM_2$

Let $M = AM_1 + BM_2 > 0$

Hence h=f.g is of bounded variation on [a,b]

From equation (3)

$\sum_{k=1}^{n} |\Delta h_k| \leq A \sum_{k=1}^{n} |\Delta f_k| + B \sum_{k=1}^{n} |\Delta g_k|$

Taking supremum, we get

$\Rightarrow V_h \leq A\, V_f + B\, V_g$

(i.e.) $V_{f.g} \leq A\, V_f + BV_g$

Note:

If f and g are each of bounded variation on [a, b], then f/g need not be of bounded variation on [a, b]

For, if $f(x) \to 0$ as $x \to x_0$, then 1/f will not be bounded on any interval containing

⇒1/f cannot be of bounded variation on such an interval [by Theorem1.8]

⇒1/f need not be of bounded variation

Theorem 1.13:

let *f* be of bounded variation on [a, b] and assume that 'f' is bounded away from zero; (i.e.) Suppose that there exists a positive number 'm' such that $o<m\leq|f(x)|$ for all a in [a, b]. Then g=1/f is also of bounded variation on [a,b], and $V_g \leq V_f/m^2$

Proof:

Let *f* be of bounded variation on [a, b]

⇒there exist a positive number K>0 such that for all partition

$P= \{a =x_1, x_2, \ldots . x_n=b\}$,

we have $\sum_{k=1}^{n} |\Delta h_k|<K$

$\sum_{k=1}^{n} | f(x_k) - f(x_{k-1}) |\leq K$

Assume *f* is bounded away from zero

(i.e.) there exist a positive number 'm' such that $0<m\leq|f(x)|$ for all $x\in[a,b]$

$\Rightarrow \frac{1}{|f(x)|} \leq \frac{1}{m} < 0$ for all $x\in[a,b]$

To prove that: 1/f is of bounded variation

Let g=1/f

Now,

$$\sum_{k=1}^{n} |\Delta g_k| \leq \sum_{k=1}^{n} |g(x_k)- g(x_{k-1})|$$

$$= \sum_{k=1}^{n} |\tfrac{1}{f}(x_k)- \tfrac{1}{f}(x_{k-1})|$$

$= \sum_{k=1}^{n} | \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})}|$

$= \sum_{k=1}^{n} | \frac{f(x_{k-1}) - f(x_k)}{f(x_k) f(x_{k-1})}|$ …………..(1)

$\leq K/m^2$

Let $M = K/m^2 > 0$

$\therefore \sum_{k=1}^{n} |\Delta g_k| \leq M$

$\therefore g = 1/f$ is of bounded variation on [a, b]

From 1, $\sum_{k=1}^{n} |\Delta g_k| \leq 1/m^2 \sum_{k=1}^{n} |\Delta f_k|$

Taking supremum, we get,

$V_g \leq 1/m^2 . V_f$

(i.e.) $V_g \leq V_f/m^2$

Additive property of Total variation

Theorem 1.14:

Let *f* be of bounded variation on [a b] and assume that $c \in [a,b]$. Then f is of bounded variation on [a, c] and on [c, b] and we have

$V_f (a, b) = V_f (a, c) + V_f (c, b)$.

Proof:

Let *f* be of bounded variation on [a, b] ……………(1)

Let $c \in (a,b)$

First, to prove *f* is of bounded variation on [a, c] & on [c, b]

Let $P_1 = \{ a = x_0, x_1, ..., x_n = c\} \in \mathcal{P}$ [a, c]

$P_2 = \{ c = y_0, y_1, ...y_m = b\} \in \mathcal{P}$ [c,b]

Then $P = P_1 U P_2 = \{ a = x_0, x_1, ..., x_n = c == y_0, y_1, ...y_m = b\} \in \mathcal{P}[a,b]$

From equation (1), we get,

$\sum(P) \leq M$ for all $M>0$

Now, Clearly,

$\sum(P_1) \leq \sum(P)$ & $\sum(P_2) \leq \sum(P)$

$\sum(P_1) \leq M$ & $\sum(P_2) \leq M$ for all $M>0$

$\therefore$ *f* is of bounded variation on [a, c] & on [c, b]

Second, to prove: $V_f(a, b) = V_f(a, c) + V_g(c, b)$

(i.e.) To Prove: $V_f(a, b) \leq V_f(a, c) + V_f(c, b)$ & $V_f(a, b) \geq V_f(a, c) + V(c, b)$

Now, We can write

$\sum(P_1) + \sum(P_2) = \sum(P) \leq V_f(a, b)$

$\sum(P_1) + \sum(P_2) \leq V_f(a, b)$

$V_f(a, c) + V_f(c, b) \leq V_f(a,b)$ ……………(3)

To obtain the reverse inequality,

Let $P=\{x_0,x_1,x_2,\ldots.x_n\} \in \mathcal{P}[a,b]$

$\because$ P is a partition of [a,b] & $c \in [a,b]$, then there exist k such that $x_{k-1} \leq c \leq x_k$

Define a new partition

$P_0 = \{ a = x_0,x_1,x_2,\ldots.x_{k-1},c,x_k,\ldots\ldots x_n=b\} \in \mathcal{P}[a,b]$

Let $P_1 = \{ a = x_0,x_1,x_2,\ldots.x_{k-1},c\} \in \mathcal{P}[a, c]$ be a partition on [a, c] &

Let $P_2 = \{c,x_k,x_{k+1},\ldots\ldots x_n=b\} \in \mathcal{P}[c, b]$ be a partition on [c, b]

Then $P_0=P_1 U P_2$

Now, we have

$\sum(P) \leq \sum(P_0)$

$= \sum(P_1) + \sum(P_2)$

(i.e.) $\sum(P) \leq \sum(P_1) + \sum(P2)$

Taking supremum, we get

$V_f(a, b) \leq V_f(a, c) + V_f(c, b)$ ……………(4)

From equation (3) & (4),

$V_f(a, b) = V_f(a, c) + V_f(c, b)$

Total Variation on [a, x] as a function of "x".

Theorem 1.15.

Let *f* be of bounded variation on [a, b]. Let V be defined on [a, b] as follows:

$V(x) = V_f(a, x)$ if $a<x \leq b$, $v(a) = 0$.

Then, (i) V is an increasing function on [a, b]

(ii) V-f is an increasing function on [a, b]

Proof:

Let *f* be of bounded variation on [a, b]

Let V: $[a,b] \rightarrow \mathbb{R}$ such that

$$V(x) = \begin{cases} 0 & \text{if } x = a \\ V_f(a, x) & \text{if } a < x \leq b \end{cases} \qquad \text{…………(1)}$$

(i)To prove: V is an increasing function on [a, b]

Let $x, y \in [a,b]$

Let $a < x < y \leq b$

To prove: $V(x) \leq V(y)$

Now, $a<x<y$

$\Rightarrow V_f(a, y) = V_f(a, x) + V_f(x, y)$ [$\because$ by Theorem 1.14]

$\Rightarrow V(y) = V(x) + V_f(x, y)$ [$\because$ by equation (1)]

$\Rightarrow V(y) - V(x) \geq V_f(x, y)$ [$V_f \geq 0$]

(i.e.) $V(y) - V(x) \geq 0$

$\Rightarrow V(x) \leq V(y)$

V is an increasing function on [a, b].

ii) To prove: V-f is an increasing function on [a, b]

Let x, y$\in$ [a,b] & $a<x<y \leq b$

To prove: (V-f) (x) $\leq$(V-f) (y)

(i.e.) To prove V(x) - f(x) $\leq$ V(y) - f(y)

(i.e.) To prove f(y)- f(x) $\leq$ v(y)-V(x)

Now, V(y) - V(x) = V_f (a, y) - V_f(a, x)

$= V_f (x, y)$ [$\because$ a<x<y [V_f (a, y) = V_f(a, x) + V_f (x, y)]

(i.e.) V(y) - V(x) =V_f (x, y)

Consider the partition p={x, y} $\in$ P[x, y]

This is the smallest partition on [x, y]

$\therefore$ V_f ({x, y}) = |f(y) - f(x)|

Now, we know that V_f(x, y) $\geq$ V_f({XY})

$\Rightarrow$ V(y)-V(x) $\geq$ |f(y)-f(x)|

$\geq$f(y) - f(x)

(i.e.) V(y)-V(x)$\geq$f(y) - f(x)

$\therefore$(V-f) (x) $\leq$(V-f) (y)

Hence V-f is an increasing function on [a, b]

Note 1.16:

Let g$\in\mathbb{R}$ on [a,b] and define f(x) $=\int_a^x g(t)\,dt$ if x$\in$[a,b]

Then the integral $\int_a^x |g(t)|\,dt$ is the total variation of 'f' on [a, x].

[(i.e.) For some functions f, the total variation V_f (a, x) can be expressed as an integral.]

Proof:

Let g∈ R

Let f(x) = $\int_a^x g(t)$dt if x∈ [a,b]

To prove: $\int_a^x |g(t)|$dt is the total variation of `f' on [a, x]

(i.e.)To prove: $V_f(a, x) = \int_a^x |g(t)|$dt

Functions of Bounded Variation Expressed as the Difference of Increasing Functions

Theorem 1.17:

Let f be defined on [a, b]. Then f is of bounded variation on [a, b] if and if only f can be expressed as the difference of two increasing functions.

Proof:

let f be defined on [a, b]

Let f be of bounded variation on [ab]

Let V: [a, b] → ℝ

$$V(x)=\begin{cases} 0 & if\ x = a \\ V_f(a,x) & if\ a < x \le b \end{cases}$$

Then, we write

f(x) = V(x) - [V(x) - f(x)]

f(x) = V(x) -(V-f) (x)

By Theorem 1.15,

V and V-f are both increasing functions on [a, b]

∴ f can be expressed as the difference of two Increasing functions.

Conversely,

Suppose that f can be expressed as the difference of two increasing functions.

Let $f = f_1 - f_2$ where f_1 & f_2 are increasing functions on [a, b]

$\because f_1$ & f_2 are increasing function on [a, b] & by Theorem1.6

f_1 & f_2 are of bounded variation on [a, b]

$\Rightarrow f_1$ & f_2 is of bounded variation on [a, b]   [$\because$ by Theorem1.11]

(i.e.) f is a bounded variation on [a, b]

Note 1.18:

The above Theorem 1:17 holds if "Increasing" is replaced by strictly increasing"

The representation of a function of bounded variance as a difference of two increasing functions is by no means unique.

If $f = f_1 - f_2$ where f_1 &f_2 are increasing,

then $f = (f_1+g) - (f_2+g)$ where g is an arbitrary increasing function

We get a new representation of f

If 'g' is strictly increasing, then the same will be true of f_1+g and f_2+g. Hence the Theorem1.17 holds if "increasing" is replaced by "strictly increasing".

Continuous Functions of Bounded Variation.

Theorem 1.19:

Let f be of bounded variation on [a, b]. If $x \in (a,b]$, let $V(x) = V_f(a, x)$ and put $V(a)=0$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.

Proof:

Let f be of bounded variation on [a, b]

Let $V(x) = \begin{cases} 0 & if\ x = a \\ V_f(a,x) & if\ a < x \leq b \end{cases}$

Case(i)

Assume that 'V' is a continuous function

Given: $\varepsilon>0$, there exist $\delta>0$ such that $|y-x|<\delta \Rightarrow |V(y)-V(x)|<\varepsilon$

To prove: f is a continuous function

By Theorem 1.15,

$\Rightarrow$V & V-f is a monotonic increasing function

$\Rightarrow V(x+)$ & $V(x-)$ exist for each $x\in (a,b)$

By Theorem 1.17,

f is monotonic

$\Rightarrow f(x+)$ and $f(x-)$ exist for each $x \in (a,b)$.

Let $a<x < y \leq b$.

Then by the definition of $V_f(x, y)$

$0\leq|f(y)-f(x)| \leq V_f(x, y)$

$\leq V_f(a, y) - V_f(a, x)$

$= V(y)-V(x)$

$\therefore 0\leq |f(y) - f(x)| \leq V(y) - V(x)$

$\leq |V(y) - V(x)|$

$\leq \varepsilon$

$|f(y) - f(x)| \leq \varepsilon$

$\therefore$ *f* is a continuous function

$\therefore$ A point of continuity of 'V' is a point of continuity of *f*.

Case(ii)

Given f is a continuous function.

To prove, V is a continuous function

Let f be continuous at $c\in (a,b)$ c

$\Rightarrow$given $\varepsilon>0$ and $\delta>0$ such that $|x-c|<\delta$

$\Rightarrow|f(x)-f(y)|<\varepsilon/2$ ....(1)

For this 'ε', there exist a partition P of [c, b], say

P={c=x_0,x_1,….x_n=b} such that $\sum_{k=1}^{n}|\Delta f_k|$>$V_f$(c, b)-ε/2 ………(2)

Assume that 0<|x_1-x_0|< δ

⇒ [f(x_1)- f(x_0)| < ε/2 [f is continuous] ……..(3)

From equation (2)

V_f(c, b) - ε/2 <$\sum_{k=1}^{n}|\Delta f_k|$

= |Δf_1|+$\sum_{k=1}^{n}|\Delta f_k|$

= |f(x_1) - f(x_0) +$\sum_{k=1}^{n}|\Delta f_k|$

< ε/2 + V_f(x_1,b) [∵by equation (3) & Definition of V_f]

V_f(c, b) - ε/2 < ε/2 + V_f(x_1,b)

V_f(c, b)-V_f(x_1,b) < ε/2 +< ε/2

< ε

V_f(c, b)-V_f(x_1,b) < ε ………….(4)

Let |x_1-c|<δ

Now,

V(x_1)-V(c) = V_f (a, x_1) - V_f (a, c)

= V_f (c, x_1)

= V_f (c, b) - V_f (x_1, b)

< ε (by equation (4))

∴V(x_1)-V(c)< ε

(i.e.)|x_1-c|< δ ⇒ |V(x_1)-V(c)|<ε

∴V(c+) = V(c)

Similarly, V(c-)= V(c)

∴ V is continuous at a point c

Hence V is a continuous function.

Thus a point of continuity of *f* is a point of continuity of *V*.

Theorem 1.20:

Let *f* be continuous on [a, b]. Then *f* is of bounded variation on [a, b] if and only if *f* can be expressed as the difference of two increasing continuous functions.

Proof:

Combining Theorem 1.19 with 1.17, we can state 1.20.

Absolute and Conditional Convergence

Definition 1.21:

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges. It is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 1.22:

Absolute convergence of $\sum a_n$ implies convergence

Proof:

Given $\sum a_n$ is absolute convergence

(i.e.) $\sum |a_n|$ converges

To prove: $\sum a_n$ converges

By Cauchy's condition for convergent series,

[The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all ε>0, there exist N∈N such that if n ≥ N then

$|a_{n+1}, a_{n+2}, \ldots\ldots, a_{n+p}| < \varepsilon$ for each p = 1,2,3,……]

For all ε>0, there exist N∈N such that if n ≥ N then

$|a_{n+1}|+|a_{n+2}|+\ldots\ldots+|a_{n+p}| < \varepsilon$ for each p = 1,2,3,…… ……. (1)

We know that,

$|a_{n+1}+a_{n+2}+\ldots\ldots +a_{n+p}| \leq |a_{n+1}|+|a_{n+2}|+\ldots\ldots +|a_{n+p}|$

$\leq ||a_{n+1}|+|a_{n+2}|+\ldots\ldots +|a_{n+p}||$

$\leq \varepsilon$ (by equation (1))

$\therefore |a_{n+1}+a_{n+2}+\ldots\ldots+a_{n+p}| < \varepsilon$

Hence $\sum a_n$ converges.

Note 1.23:

The converse of the above Theorem is not true.

For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is the convergent series.

But $\sum_{n=1}^{\infty} |\frac{(-1)^{n+1}}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is not a convergent series.

Theorem 1.24:

Let $\sum a_n$ be a given series with real-valued terms and define $p_n=\frac{|a_n|+a_n}{2}$, $q_n=\frac{|a_n|-a_n}{2}$. (n=1,2 ……) Then

(i) If $\sum a_n$ is conditionally convergent, both $\leq \sum p_n$ & $\sum q_n$ diverge

(ii) If $\sum |a_n|$ converges, both $\sum p_n$ and $\sum q_n$ converges and we have

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$

Proof:

Let $\sum a_n$ be a given series with real-valued terms

Define $p_n=\frac{|a_n|+a_n}{2}$, $q_n=\frac{|a_n|-a_n}{2}$

$\Rightarrow P_n = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases}$ & $q_n = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases}$ ………..(1)

$\Rightarrow P_n \geq 0$ & $q_n \geq 0$

Also $p_n - q_n = a_n$ & $p_n+q_n = |a_n|$ ……. (2)

(i) Given $\sum a_n$ is conditionally convergent

$\Rightarrow \sum a_n$ converges but $\sum |a_n|$ diverges ……..(3)

To prove: $\sum p_n$ & $\sum q_n$ diverge

If $\sum p_n$ converges,

$\Rightarrow \sum q_n$ converges ($\because q_n = p_n - a_n$)

If $\sum q_n$ converges,

$\Rightarrow p_n$ converges ($\because p_n = q_n + a_n$)

Hence if $\sum p_n$ & $\sum q_n$ converges, both $\sum p_n$ & $\sum q_n$ converges

$\Rightarrow \sum p_n + \sum q_n$ converges

[by Theorem1.22, let $\sum a_n$ and $\sum b_n$ converges. Then $\sum(\alpha a_n + \beta a_n)$ converges. For all α, β & $\sum(\alpha a_n + \beta a_n) = \alpha \sum a_n + \beta \sum b_n$]

which is a contradiction to equation (3)

Hence $\sum p_n$ & $\sum q_n$

(ii) Given $\sum |a_n|$ converges

To prove: $\sum p_n$ & $\sum q_n$ converge and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$

$\sum |a_n|$ converges

$\Rightarrow \sum(p_n + q_n)$ converges

$\Rightarrow \sum p_n + \sum q_n$ converges

$\Rightarrow \sum p_n \& \sum q_n$ converges

Now, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(p_n - q_n)$

[by Theorem 1.22, let $\sum p_n$ and $\sum q_n$ converges. Then $\sum(\alpha p_n + \beta q_n)$ converges.

For all α, β & $\sum(\alpha p_n + \beta q_n) = \alpha \sum p_n + \beta \sum q_n$]

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$

Dirichlet's Test and Abel's Test

Theorem 1.25:

If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex numbers, define

$A_n = a_1+a_2+\ldots\ldots +a_n$. Then we have the identity

$\sum_{n=1}^{\infty} a_k b_k = A_nb_{n+1} - \sum_{n=1}^{\infty} A_k (b_{k+1)} - b_k)$

Therefore $\sum_{n=1}^{\infty} a_k b_k$ converges if both the series. $\sum_{n=1}^{\infty} A_k (b_{k+1)} - b_k)$ and the sequence $\{A_nb_{n+1}\}$ converge.

Proof:

Let $\{a_n\}$ & $\{b_n\}$ be two sequences of complex numbers

Define $A_n=a_1 +a_2+\ldots. +a_n$

$a_k = A_k-A_{k-1}$

Let $A_0 = 0$

Now,

$$\sum_{n=1}^{\infty} a_k b_k = \sum_{k=1}^{n} [A_k - A_{k-1}] b_k$$

$$= \sum_{k=1}^{n} A_k b_k - \sum_{n=1}^{\infty} A_{k-1} b_k$$

$$= \sum_{k=1}^{n} A_k b_k - \sum_{n=1}^{\infty} A_k b_{k+1} - A_0b_1 + A_nb_{n+1}$$

$$= \sum_{k=1}^{n} A_k (b_k - b_{k+1}) + A_nb_{n+1}$$

$\therefore \sum_{n=1}^{\infty} a_k b_k = A_nb_{n+1} - \sum_{n=1}^{\infty} A_k (b_{k+1} - b_k)$

Now,

$$\sum_{n=1}^{\infty} a_k b_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k$$

$$= \lim_{n\to\infty} [A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$$

$\sum_{n=1}^{\infty} a_k b_k = \lim_{n\to\infty} [A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$

$\because \{A_nb_{n+1}\}$ & $\sum_{k=1}^{\infty} A_k (b_{k+1} + b_k)$ Converge, by Theorem8.8

$\sum_{n=1}^{\infty} a_k b_k$ converges

Theorem1.26: (Dirichlet's Test)

Let $\sum a_n$ be a series of complex terms whose partial sums form a bounded sequence. Let $\{b_n\}$ be a decreasing sequence which converges to 0. Then $\sum a_n b_n$ converges

Proof:

Let $\sum a_n$ be a series of complex terms whose partial Sums form a bounded sequence.

Let $A_n = a_1+a_2+\ldots\ldots a_n$

Given $\{A_n\}$ bounded.

There exist M>0such that $|A_n| \leq M$ for all $n \in N$ ………(1)

Let $\{b_n\}$ be a decreasing sequence which converges to 0

(i.e.) $\lim_{n\to\infty} b_n = 0$ …….. (2)

From equation (1) & (2)

$\lim_{n\to\infty} A_n b_{n+1} = 0$

(i.e.) $\{A_n b_{n+1}\}$ converges ……… (3)

It is enough to prove that $\sum A_n (b_{n+1} - b_n)$ converges

Now,

$\sum_{n=1}^{\infty} A_n(b_{n+1} - b_n) = \sum_{n=1}^{\infty} |A_n| (b_{n+1} - b_n)$

$\leq \sum_{n=1}^{\infty} M.(b_{n+1} - b_n)$ converges

$= M.\sum_{n=1}^{\infty}(b_n - b_{n+1})$

$\sum_{n=1}^{n} A_n(b_{n+1} - b_n) \leq M.\sum_{n=1}^{n}(b_n - b_{n+1})$

Let $S_n = \sum_{n=1}^{\infty}(b_k - b_{k+1}) = b_1 - b_{n+1}$

$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} b_1 - b_{n+1} = b_1 - 0 = b_1$

$\therefore \sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges

$\Rightarrow M.\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges

By comparison test, we have

$\sum_{n=1}^{\infty} A_n (b_{n+1} - b_n)$ converges ………... (4)

From equation (3) & (4)

$\therefore A_n b_{n+1}\}$ & $\sum A_n (b_{n+1} - b_n)$ converges

By Theorem 1.25,

$\sum a_n b_n$ converges.

Theorem 1.27:(Abel's Test)

The series $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\{b_n\}$ is a monotonic convergent sequence.

Proof:

Given $\sum a_n$ converges and $\{b_n\}$ is a monotonic convergent sequence.

Let $A_n = a_1 + a_2 + \ldots + a_n$

$\because \sum a_n$ converges, then the sequence of partial sums

$\lim_{n\to\infty} A_n$ converges

$\Rightarrow$This sequence $\{A_n\}$ bounded

$\Rightarrow M>0$ such that : $|A_n| < M$ for every $n \in N$ ……… (1)

$\because \{b_n\}$ is a convergent sequence & by equation (1),

$\lim_{n\to\infty} A_n\, b_{n+1}$ converges ………. (2)

Now,

$$\sum_{n=1}^{\infty} |A_n(b_{n+1}-b_n)| = \sum_{n=1}^{\infty} |A_n|.|(b_{n+1}-b_n)|$$

$$\leq \sum_{n=1}^{\infty} |M.|(b_{n+1}-b_n)|$$

$$\leq M \sum_{n=1}^{\infty} |(b_{n+1}-b_n)|$$

$$\sum_{n=1}^{\infty} |A_n(b_{n+1}-b_n)| \leq M \sum_{n=1}^{\infty} |(b_{n+1}-b_n)| \qquad \text{……….(3)}$$

Case(i)

$\{b_n\}$ is a monotonically increasing sequence

$$\sum_{n=1}^{\infty} |\, b_{n+1} - b_n| = \sum_{n=1}^{\infty}(b_{n+1}-b_n)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{n=1}^{\infty} |(b_{n+1}-b_n)| = -b_1$$

(i.e.) $\sum_{n=1}^{\infty} |(b_{n+1}-b)|$ converges to $-b_1$

Case (ii)

$\{b_n\}$ is a monotonically decreasing sequence.

$$\sum_{n=1}^{\infty} |\, b_{n+1} - b_n \,| = \sum_{n=1}^{\infty}(b_{n+1}-b_n)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{n=1}^{\infty} |(b_{n+1}-b_n)| = b_1$$

(i.e.) $\sum_{n=1}^{\infty} |(b_{n+1}-b_n)|$ converges to b_1

In either case we see that the series

$\sum_{n=1}^{\infty} |\, b_{n+1} - b_n|$ converges $\Rightarrow$ M. $\sum_{n=1}^{\infty} |\, b_{n+1} - b_n|$ converges.

By comparison Test,

$\sum_{n=1}^{\infty} |A_n(b_{n+1}-b_n)|$converges.

$\Rightarrow \sum_{n=1}^{\infty} |A_n(b_{n+1}-b_n)|$converges ………..(3)

From equation (2) and (3), by Theorem 1.25,

$\sum a_n b_n$ converges.

Rearrangements of Series

Definition 1.28:

Let *f* be a function whose domain is Z^+ and whose range is Z^+, and assume that 'f' is one-one on Z^+. Let $\sum a_n$ and $\sum b_n$ be two series such that $b_n = a_{f(n)}$ for n=1,2,….

Then $\sum b_n$ is said to be a rearrangement of $\sum a_n$.

($Z^+ = \{1,2,3,...\}$, $b_n = a_{f(n)} \Rightarrow a_n = b_{f^{-1}(n)}$)

$\sum a_n$ is also a rearrangement of $\sum b_n$

Theorem1.29:

Let $\sum a_n$ be an absolutely convergent series having sum S. Then every rearrangement of $\sum a_n$ also converges absolutely and has sum S.

Proof:

Let $\sum a_n$ be an absolutely convergent series having Sum 'S'

(i.e.) $\sum a_n$ converges & $|\sum a_n| = S$ ……. (1)

Let f: $Z^+ \to Z^+$ & assume that f is 1-1

Let $\sum b_n$ be a rearrangement of $\sum a_n$

Define $b_n = a_{f(n)}$ for every n= 1,2,3,… ……… (2)

To prove: $\sum b_n$ converges absolutely and $\sum b_n = S$

(i.e.) To prove: $\sum |b_n|$ converges & $\sum b_n = S$

Now,

$|b_1| + |b_2| + \ldots + |b_n| = |a_{f(1)}| + |a_{f(2)}| + \ldots + |a_{f(n)}|$

$= |a_1| + |a_2| + \ldots + |a_n|$

$\leq |a_1| + |a_2| + \ldots + |a_n|$

$= \sum_{n=1}^{\infty} |a_n| = S$ (by 1)

$\therefore |b_1| + |b_2| + \ldots |b_n| < S$

(i.e.) $\sum |b_n|$ has bounded partial sum

$\therefore \sum |b_n|$ converges

(i.e.) $\sum b_n$ converges absolutely

Now, to prove $\sum b_n = S$

Let $t_n = b_1 + b_2 + \ldots + b_n$ & $S_n = a_1 + a_2 + \ldots + a_n$

Given $\varepsilon > 0$, choose N so that $|S_N - S| < \varepsilon/2$ ..……….(3)

$||S_N|-|S|| = |S_N-S| < \varepsilon/_2$

$\Rightarrow ||a_1|+|a_2|+.....+|a_N|-(|a_1|+|a_2|+....+|a_N|+|a_{N+1}|+.....)|<\varepsilon/2$

$\Rightarrow |-(|a_{N+1}|+|a_{N+2}|+.......)| <\varepsilon/2$

$\Rightarrow \sum_{K=1}^{\infty} |a_{N+K}| <\varepsilon/2$ ....(4)

Now,

$|t_n -S| = |t_n -S_N + S_N-S|$

$\leq |t_n-S_N|+ |S_N-S|$

$\therefore |t_n -S|<|t_n-S_N|+ \varepsilon/2$ (by 3) ....(5)

Choose M so that $\{1, 2 ...., N\} \subseteq \{f(1)\ f(2), ...,f(M)\}$

Then $n> M \Rightarrow f(n) > N$, and for such 'n' we have

$|t_n - S_N| = |b_1+ b_2+...... + b_n - (a_1 +a_2+ ......a_N) |$

$= |a_{f(1)} + a_{f(2)} +........+ a_{f(n)}| - (a_1+a_2+.....+a_N)|$

$= |a_1+a_2+......+a_n - (a_1 +a_2 +.......+a_N)|$

$= |a_{N+1}+a_{N+2}+....... +a_n|$

$= |a_{N+1}| +|a_{N+2}|+......+|a_N|$

$\leq |a_{N+1}| +|a_{N+2}|+......$

$= \sum_{K=1}^{\infty} |a_{N+K}|$

$< \varepsilon/2$ (by 4)

$\therefore |t_n - S_N|< \varepsilon/2$ ....(6)

Now, (5) $\Rightarrow |t_n - S_N|<|t_n-S_N|+ \varepsilon/2$

$< \varepsilon/2+ \varepsilon/2$ (by equation (6))

$= \varepsilon$

$\therefore |t_n - S_N|< \varepsilon$

Hence for all ε>0, there exist M such that $|t_n - S_N| < \varepsilon$ for all n>M

$\lim_{n\to\infty} t_n = S$

Riemann's Theorem on Conditionally Convergent Series:

Theorem 1.30:

Let $\sum a_n$ be a conditionally convergent series with real-valued terms. Let x and y be given numbers in the closed interval [-∞,+∞], with x≤y. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that $\lim_{n\to\infty}$ inf t_n = x and $\lim_{n\to\infty} t_n$ sup t_n = y,

where $t_n = b_1+b_2+\ldots\ldots+b_n$

Proof:

Let $\sum a_n$ be a conditionally convergent series with real-valued terms.

Let $-\infty \le x \le y \le \infty$

Discarding those terms of a series which are zero does not affect its convergence or divergence.

Hence we might as well assume that no terms of $\sum a_n$ are zero

Let P_n denote the n^{th} positive term of $\sum a_n$ &

Let $-q_n$ denote the n^{th} negative term of $\sum a_n$.

Define $p_n = \frac{|a_n|+a_n}{2}$ & $q_n = \frac{|a_n|-a_n}{2}$ (n=1,2,3,…….)

$\Rightarrow p_n = \begin{cases} a_n & if\ a_n \ge 0 \\ 0\ if\ a_n < 0 \end{cases}$ and $\Rightarrow q_n = \begin{cases} -a_n & if\ a_n \le 0 \\ 0\ if\ a_n > 0 \end{cases}$

$\Rightarrow p_n \ge 0$ and $q_n \ge 0$

$\Rightarrow p_n + q_n = |a_n|$ & $p_n - q_n = a_n$ …… (1)

$\because \sum a_n$ is conditionally convergent,

$\sum a_n$ converges but $\sum |a_n|$ diverges ……. (2)

If $\sum p_n$ converges & by 1, $\sum q_n$ converges.

If $\sum q_n$ converges & by 1, $\sum p_n$ converges.

(i.e.) both $\sum p_n$ & $\sum q_n$ converges

$\Rightarrow \sum p_n$ & $\sum q_n$ converges

$\Rightarrow \sum |a_n|$ converges.

Contradiction to equation (2)

Hence $\sum p_n$ & $\sum q_n$ divergences.

Let $\{x_n\}$ &$\{y_n\}$ be two sequences of real numbers there exist

$\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$ with $x_n < y_n$, $y_1 > 0$

Let (k_1, r_1) be the least positive integer there exist

$p_1+p_2+\ldots\ldots+p_{K_1} > y_1$ with $y_1>0$

$p_1+p_2+\ldots\ldots+p_{K_1} - q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+p_{K_{1+2}}+\ldots\ldots.+p_{K_2} > y_2$ &

$p_1+p_2+\ldots\ldots+p_{K_1}-q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+p_{K_{2+2}}+\ldots\ldots.+p_{K_2}-q_{r_{1+1}}$

$$-q_{r_{2+2}}-\ldots\ldots.-q_{r_2} < x_2$$

These steps are possible because $\sum p_n$ & $\sum q_n$ are both divergent series of positive terms. If the process is continued in this way,

we obtain a rearrangement $\sum b_n$ & $\sum a_n$

Where $\sum b_n = p_1+p_2+\ldots\ldots+p_{K_1}-q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+p_{K_{2+2}}+\ldots\ldots.+p_{K_2}-q_{r_{1+1}}$

$$-q_{r_{2+2}}-\ldots..-q_{r_2}\ldots \qquad \ldots\ldots (3)$$

Let $t_n = b_1+b_2+\ldots\ldots +b_n$

To prove: $\lim_{n\to\infty} \inf t_n = x$ and $\lim_{n\to\infty} t_n \sup t_n = y$,

Let α_n and β_n denote the partial sum of equation (3) whose last terms are p_{K_n} & q_{r_n} respectively.

Since, $p_1+p_2+\ldots\ldots+p_{K_1} - q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+\ldots\ldots.+p_{K_{n-1}}+p_{K_n} > y_n$

$\therefore p_1+p_2+\ldots\ldots+p_{K_1} - q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+\ldots\ldots.+p_{K_{n-1}} \le y_n$

$\Rightarrow p_1+p_2+\ldots\ldots+p_{K_1}- q_1-q_2-\ldots..-q_{r_1}+p_{K_{1+1}}+\ldots\ldots.+p_{K_{n-1}}+p_{K_n}\leq y_n+p_{K_n}$

$\Rightarrow \alpha_n \leq y_n+p_{K_n}$

$\Rightarrow \alpha_n- y_n \leq p_{K_n}$

$\Rightarrow |\alpha_n- y_n| \leq p_{K_n}$ as $\alpha_n>y_n$ & $p_{K_n}>0$

Similarly,

$|\beta_n- y_n| \leq q_{r_n}$

$\because \sum a_n$ is converges, $a_n \to 0$ as $n\to\infty$

$\Rightarrow p_n \to 0$ & $q_n\to 0$ as $n\to\infty$

$\Rightarrow p_{K_n} \to 0$ & $q_{r_n}\to 0$ as $n\to\infty$

Now, $\Rightarrow p_{K_n} \to 0$ as $n\to\infty$

Given $\varepsilon>0$, there exist a positive integer N_2 such that

$| p_{K_n}| <\varepsilon/2$ for all $n\geq N_1$

Again since $y_n \to y$ as $n\to\infty$

Given $\varepsilon>0$, there exist a positive integer N_2 such that

$|y_n-y| <\varepsilon/2$ for all $n\geq N_2$

Let $N = \max \{N_1, N_2\}$, Then

$$\begin{aligned}|\alpha_n- y| &= |\alpha_n- y_n+y_n-y|\\ &\leq |\alpha_n- y_n|+|y_n-y|\\ &\leq p_{K_n}+|y_n-y|\\ &\leq | p_{K_n}|+|y_n-y|\\ &\leq \varepsilon/2 +\varepsilon/2 = \varepsilon\end{aligned}$$

$\therefore |\alpha_n- y| < \varepsilon$ for all $n \geq N$

But ‘ε’ was arbitrary, $\therefore \alpha_n \to y$

similarly, by the same argument we can prove that $\beta_n \to x$

Finally, it is clear that no number less than 'x’ or greater than 'y' can be sub sequential limit of the partial sum of equation (3)

$\lim_{n\to\infty}$ inf t_n = x and $\lim_{n\to\infty} t_n$ sup t_n = y, where $t_n = b_1+b_2+\ldots+b_n$

Unit II

The Riemann - Stieltjes Integral: Introduction - Notation - The definition of the Riemann - Stieltjes integral - Linear Properties - Integration by parts- Change of variable in a Riemann - Stieltjes integral - Reduction to a Riemann Integral – Euler's summation formula - Monotonically increasing integrators, Upper and lower integrals - Additive and linearity properties of upper, lower integrals – Riemann's condition - Comparison theorems.

2.1. Introduction

- Finding the slope of the tangent line to a curve is studied by a limit process known as differentiation
- Finding the area of a region under a curve is studied by a limit process known as integration.
- To find the area of the region under the graph of a positive function f defined on [a,b], we subdivide the interval [a,b] into a finite number of subintervals, say n, the kth subinterval having length Δx_k and the sum $\sum_{k=1}^{n} f(x_k)\Delta x_k$, where $t_k \in (x_{k-1}, x_k)$ is an approximation to the area by means of rectangles.
- If the definite integral of a continuous function f as a function of its upper limit, we write $F(x) = \int_a^x f(t)\, dt$.Then $F'(x) = f(x)$.Hence differentiation and integration are in inverse operations.

Notation 2.2:

- $f, g, \alpha, \beta \rightarrow$ Real-valued functions defined bounded on [a, b].

Definition 2.3: A partition P of [a, b], where a<b is a finite set $P = \{a = x_0, x_1, x_2, \ldots . x_n = b\}$ such that $a = x_0 \leq x_1 \leq, \ldots, \leq x_{i-1} \leq x_i \leq, \ldots, \leq x_n = b$.

- $\Delta x_k = x_k - x_{k-1}$
- $\wp[a, b]$ = set of all partitions of [a, b]
- A partition P' of [a,b] is said to be finer than (or a refinement of) if $P \subseteq p'$
- $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ and $\sum_{k=1}^{n} \Delta x_k = \alpha(b) - \alpha(a)$

- The mesh (or) norm of a partition P is the length of the largest subinterval of P. (i.e.) $\|P\| = \max_{k\epsilon\{1,2\ldots n\}} |x_k - x_{k-1}|$
- $P \subseteq P' \Rightarrow \|P\| \geq \|P'\|$

  (ie) The refinement of a partition decreases its norm, but the converse does not necessarily.

The Definition of the Riemann-Stieltjes Integral

Definition 2.4:

Let f, α be the real-valued functions defined on the closed interval [a,b]. Let $P = \{a = x_0, x_1, x_2, \ldots . x_n = b\}$

be partition of [a,b] and let be a point in the subinterval $[x_{k-1}, x_k]$. Let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$. Then a sum of the form $S(P, f, \alpha) = \sum_{k=1}^{n} f(x_k)\Delta x_k$ ……..(1)

is called a Riemann-Stieltjes Sum of f with respect to α

The function f is said to be Riemann- Stieltjes Integral with respect to α on [a,b] (i.e.) $f \in$ R(α) on [a,b]

Their exist A∈ R having the following property:

For all $\varepsilon > 0$, their exist a partition P_ε of [a,b] such that

For all partition P finer than P_ε ($P_\varepsilon \subseteq P$)

For every choice of the points $t_k \epsilon [x_{k-1}, x_k]$, we have

$|s(p, f, \alpha) - A| < \varepsilon$

(i.e.) $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = A$

If such on A∈ R exists and uniquely determined, we say that the Riemann-Stieltjes Integral exists and write

$$\int_a^b f\, d\alpha = \int_a^b f(x)\, d\alpha(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = A$$

Note 1:

The function f and α referred to as the integrand and the integrator respectively.

Note 2: Riemann Integral

When $\alpha(x) = x$ in the Riemann-Stieltjes Integral, then we get the Riemann sum of f

$S(P, f) = \sum_{k=1}^{n} f(t_k)\Delta x_k$

and $|s(p, f) - A| < \varepsilon$

Then the function f is said to be Riemann Integrable on [a, b]

(i.e.) $f \in$ R and write

$\int_a^b f\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = A$

Note:3

The numerical value of $\int_a^b f(x)\,d\alpha(x)$ depends only on $f, \alpha, a \text{ and } b$ and does not depend on the symbol x. This letter x is a dummy variable and may be replaced be any other convenient symbol.

Linear Properties

Theorem 2.5: [Linearity of the Integrand of R-S Integral]

If $f \in$ R(α) and if $g \in$ R(α) on [a,b], then $c_1 f + c_2 g \in$ R(α)

On [a, b] for any two constants c_1 and c_2 and we have

$$\int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha$$

Proof:

Let f, g, α be real valued function defined on [a, b]

Let $P = \{a = x_0, x_1, x_2, \dots . x_n = b\} \in \wp[a,b]$

$t_k \epsilon\ [x_{k-1}, x_k]\ \ and\ \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$.

Let $\varepsilon > 0$ be given

Given $f \in \mathrm{R}(\alpha)$ and $g \in \mathrm{R}(\alpha)$

Let $c_1\ and\ c_2$ be two constants

To prove$c_1 f + c_2 g \in \mathrm{R}(\alpha)$

and $\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha$

Let $h = c_1 f + c_2 g$

Now $f \in \mathrm{R}(\alpha) \Rightarrow$ their exists $A \in \mathcal{R}$ such that for all $\varepsilon > 0$ $\varepsilon_1 = \frac{\varepsilon}{2|c_1|} > 0$, their exists $P_{\varepsilon_1}$of [a,b] such that for all P finer than $P_{\varepsilon_1}\ and\ t_k \epsilon\ [x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \varepsilon_1\ and$

$A = \int_a^b f\, d\alpha \dots\dots.$ (1)

Also $g \in \mathrm{R}(\alpha) \Rightarrow$ their exists $B \in \mathcal{R}$ such that for all $\varepsilon_1 = \frac{\varepsilon}{2|c_2|} > 0$ their exists $P_{\varepsilon_2}$of [a,b] such that for all P finer than $P_{\varepsilon_2}\ and\ t_k \epsilon\ [x_{k-1}, x_k]$, we have $|S(P, g, \alpha) - B| < \varepsilon_2\ and\ B = \int_a^b g\, d\alpha \dots\dots$(2)

Let $P_\varepsilon = P_{\varepsilon_1} \cup P_{\varepsilon_2}$

Then $\forall\ P$ finer then P_ε, we have

$$|S(P, h, \alpha) - (C_1 A + C_2 B)| = \left| \sum_{k=1}^{n} h(t_k) \Delta\alpha_k - (C_1 A + C_2 B) \right|$$

$$= \left| \sum_{k=1}^{n} (C_1 f + C_2 g)(t_k) \Delta\alpha_k - (C_1 A + C_2 B) \right|$$

$$= \left| \sum_{k=1}^{n} (C_1 f)(t_k)\, \Delta\alpha_k - C_1 A + \sum_{k=1}^{n} (C_2 g)(t_k)\, \Delta\alpha_k - C_2 B \right|$$

$$= \left| C_1 \left(\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - A \right) + C_2 \left(\sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k - B \right) \right|$$

$$\leq |C_1| \left| \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - A \right| + |C_2| \left| \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k - B \right|$$

$= |C_1||S(P, f, \alpha) - A| + |C_2||S(P, g, \alpha) - B|$

$< |C_1|\varepsilon_1 + |C_2|\varepsilon_2$

$= |C_1|\frac{\varepsilon}{2|C_1|} + |C_2|\frac{\varepsilon}{2|C_2|}$ (by equation (1) & (2))

$= \varepsilon$

$\therefore |S(P, C_1 f + C_2 g, \alpha) - (C_1 A + C_2 B)| < \varepsilon$

$\therefore C_1 f + C_2 g \in R(\alpha)$

Also, $\int_a^b (C_1 f + C_2 g) d\alpha = C_1 A + C_2 B$

$$\int_a^b (C_1 f + C_2 g) d\alpha = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b g\, d\alpha$$

Theorem 2.6: [Linearity of the Integrator of R-S Integral]

If $f \in R(\alpha)$ and $f \in R(\beta)$ on [a,b], then $f \in R(C_1\alpha + C_2\beta)$ on [a, b] (for any two constants C_1 *and* C_2) and we have

$$\int_a^b f\, d(C_1\alpha + C_2\beta) = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b f\, d\beta$$

Proof:

Let f, α, β be real- valued functions defined on [a,b]

Let $P = \{a = x_0, x_1, \ldots, x_n = b\} \in \wp[a, b]$

Let $t_k \in [x_{k-1}, x_k]$ *and* $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given

Given $f \in R(\alpha)$ and $f \in R(\beta)$

Let C_1 *and* C_2 be two constants.

To prove: $f \in (C_1\alpha + C_2\beta)$ *and*

$\int_a^b f\, d(C_1\alpha + C_2\beta) = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b f\, d\beta$

Now, $f \in R(\alpha)$

$A \in \mathbb{R} \ni : \forall\, \varepsilon_1 = \frac{\varepsilon}{2|C_1|} > 0$,there exists $P_{\epsilon_1}$of [a,b] such that for all P is finer that $P_{\epsilon_1}$and $t_k \in[x_{k-1}, x_k]$ we have |S(P, f, α) – A| < ε_1 and A = $\int_a^b f$ d $f\, d\alpha$ ----(1)

Also, f ∈ $R(\beta)$

$B \in \mathbb{R} \ni : \forall\, \varepsilon_2 = \frac{\varepsilon}{2|C_2|} > 0$,there exists $P_{\epsilon_2}$of [a,b] such that for all P is finer that $P_{\varepsilon_2}$and $t_k \in[x_{k-1}, x_k]$ we have |S(P,f, β) – B| < ε_1 and A = $\int_a^b f$ d $f\, d\alpha$ ----(2)

Let $P_\varepsilon = P_{\varepsilon_1} \cup P_{\varepsilon_2}$

Then for all P finer than P_ε, we have

$|S(P, f, C_1\alpha + C_2\beta) - (C_1A + C_2B)|$

$$= \left|\sum_{k=1}^{n} f(t_k)\Delta(C_1\alpha + C_2\beta)_k - (C_1A + C_2B)\right|$$

$$= \left|\sum_{k=1}^{n} f(t_k)(C_1\Delta\alpha_k + C_2\Delta\beta_k) - (C_1A + C_2B)\right|$$

$$= |C_1(\textstyle\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A + C_2(\sum_{k=1}^{n} f(t_k)\Delta\beta_k - B|$$

$$= |C_1| \textstyle\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A + C_2 \sum_{k=1}^{n} f(t_k)\Delta\beta_k - B|$$

$$= |C_1||\, S(P, f, \alpha) - A| - |C_2||\, S(P, f, \beta) - B|$$

$$= |C_1|\varepsilon_1 + |C_2|\varepsilon_2 \quad \text{(by (1) and (2))}$$

$$= |C_1|\frac{\varepsilon}{2|C_1|} + |C_2|\frac{\varepsilon}{2|C_2|} = \varepsilon$$

$|S(P, f, C_1\alpha + C_2\beta) - (C_1A + C_2B)| < \varepsilon$

Therefore $f \in R(C_1\alpha + C_2\beta)$

Also $\int_a^b f\, d\,(C_1\alpha + C_2\beta) = C_1A + C_2B$

$\int_a^b f\, d\,(C_1\alpha + C_2\beta) = C_1 \int_a^b f d\alpha + C_2 \int_a^b f d\beta$

Theorem 2.7: [R-S Integrability on Subintervals]

Assume that c ϵ (a,b). If two of the three integrals in (1) exists, then the third also exists and we have $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$

(i.e.) Let f and α be functions on the interval [a,b] and Let c ϵ (a,b). Then (a) $\int_a^b f d\alpha$ exists if both $\int_a^c f d\alpha$ & $\int_c^b f d\alpha$ exists

(b) $\int_a^c f d\alpha$ exists if both $\int_a^b f d\alpha$ & $\int_c^b f d\alpha$ exists

(c) $\int_c^b f d\alpha$ exists if both $\int_a^b f d\alpha$ & $\int_a^c f d\alpha$ exists.

Proof:

Assume that c ϵ (a,b).

Suppose $\int_a^c f d\alpha$ and $\int_c^b f d\alpha$ exists

To prove: $\int_a^b f d\alpha$ exists and $\int_a^b f d\alpha$ $\int_a^c f d\alpha + \int_c^b f d\alpha$

Let P = {a = $x_0, x_1, \ldots, x_{k-1}, x_k = c, x_{k+1}, \ldots, x_{n-1}, x_n$ = b}

Be a partition on [a,b] Let P' = { a = $x_0, x_1, \ldots, x_{k-1}, x_k = c$}& P''= { c = $x_k, x_{k+1}, \ldots, x_{n-1}, x_n$ = b} be the partitions of [a, c] and [c, b] respectively.

The Riemann - Stieltjes sums for these partitions are connected by the equation.

$|S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha)|$----(1)

Let $\varepsilon > 0$ be given

Let $\int_a^c f d\alpha =$ A and $\int_c^b f d\alpha =$ B, A,B $\in \mathbb{R}$

Now, $\int_a^c f d\alpha =$ A $\int_c^b f d\alpha =$ B , A,B $\in \mathbb{R}$

Now, $\int_a^c f d\alpha = \text{A}$

$\Rightarrow \ \varepsilon_1 = \frac{\varepsilon}{2} > 0$, there exists $P_{\varepsilon_1}\ of\ [a, c]$ such that

For all P' finer than P_{ε_1} we have

$|S(P', f, \alpha) - A| < \varepsilon_1$ and $A = \int_a^c f d\alpha$ ……..(2)

Also $\int_c^b f d\alpha$ =B

$\Rightarrow \ \varepsilon_2 = \frac{\varepsilon}{2} > 0$, there exists $P_{\varepsilon_2}\ of\ [c, b]$ such that

For all P' finer than P_{ε_2} we have

$|S(P'', f, \alpha) - A| < \varepsilon_2$ and $B = \int_c^b f d\alpha$ ……..(3)

Let $P_\varepsilon = P_{\varepsilon_1} \cup P_{\varepsilon_2}$

Then for all P finer than P_ε we have

$|S(P, f, \alpha) - (A + B)| = |S(P', f, \alpha) + S(P'', f, \alpha) - (A + B)|$

(by equation (1))

$= |(S(P', f, \alpha) - A) + (S(P'', f, \alpha) - B)|$

$\leq |S(P, f, \alpha) - A| + |S(P'', f, \alpha) - B|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

$|S(P, f, \alpha) - (A + B)| < \varepsilon$

$\int_a^b f d\alpha$ exists

Also $\int_a^b f d\alpha = A + B$

$\Rightarrow \int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$

Similarly, we can prove the remaining two cases.

Note:

Using mathematical induction, we can prove the above result for a decomposition of [a, b] into a finite number of subintervals.

Definition 2.8.

If a < b, we define $\int_b^a f d\alpha = -\int_a^b f d\alpha$ whenever $\int_a^b f d\alpha$ exists.

We also define $\int_a^a f d\alpha = 0$

Then the equation

$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$ becomes

$$\int_a^c f d\alpha + \int_c^b f d\alpha = -\int_a^b f d\alpha$$

$$\int_a^b f d\alpha + \int_a^c f d\alpha + \int_c^b f d\alpha = 0$$

Integration by parts

Note:

A remarkable connection exists between the integrand and the integrator in the R-S integral.

The existence of $\int_a^b f d\alpha$ implies the existence of $\int_a^b \alpha\, df$ and the converse is also true.

Theorem 2.9:

[The formula of integration by parts of R-S Integral]

If $f \epsilon\ R(f)$ on [a,b] , then $\alpha \epsilon\ R(f)$ on [a,b] and we have

$$\int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

Proof:

Let f and α be real valued functions on [a,b]

Let $P = \{a = x_0, x_1, \dots, x_n = b\} \in \wp[a,b]$

Let $t_k \in [x_{k-1}, x_k]$ and $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given $f \epsilon\ R(\alpha)$

To Prove: $\alpha \epsilon\ R(f)$ and $\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$

Now, $f \epsilon\ R(\alpha)$

⇒ their exists A∈ ℝ such that for all $\varepsilon > 0$ P_ε of [a, b] such that

∀ p finer than P_ε $t_k \in [x_{k-1}, x_k]$, we have

$|S(P, f, \alpha) - B| < \varepsilon$ and $B = \int_a^b f d\alpha$

Let A= $f(b)\alpha(b) - f(a)\alpha(a)$

$\Rightarrow A = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1})$

∀ p finer than P_ε, we have

$|S(P, \alpha, f) - (A - B)| = |\sum_{k=1}^{n} \alpha(t_k)\Delta f_k - A + B|$

$\leq |\sum_{k=1}^{n} \alpha(x_k)\Delta f_k - A + B|$ $\quad (\because\ t_k \in [x_{k-1}, x_k]\,)$

$\leq |\sum_{k=1}^{n} \alpha(x_k)\,[f(x_k) - f(x_{k-1})] - \sum_{k=1}^{n} f(x_k)\alpha(x_k) +$
$\sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1})|$ +B

$\leq |\sum_{k=1}^{n} \alpha(x_k) f(x_k) - \sum_{k=1}^{n} \alpha(x_k) f(x_{k-1}) - \sum_{k=1}^{n} f(x_k)\alpha(x_k) +$
$\sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1})|$ +B

$\leq |\sum_{k=1}^{n} f(x_{k-1})[\alpha(x_k) - \alpha(x_{k-1}) - B|$

$\leq |\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - B|$

$= |S(P, f, \alpha) - B|$

$< \varepsilon$

$\therefore |S(P, \alpha, f) - (A - B)| < \varepsilon$

(i.e.) $\alpha \epsilon\ R(f)$ and $\int_a^b \alpha\, df = A - B$

$$\Rightarrow \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f d\alpha$$

$$\Rightarrow \int_a^b f d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a)$$

Change of Variable in a Riemann-Stieltjes Integral

Theorem 2.10:

Let $f \in$ R (α) on [a,b] and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that a=g(c) and b=g(d). Let h and β be the composite functions defined as follows: $h(x) = f[g(x)], \beta(x) = \alpha[g(x)]\ if\ x \epsilon S$

Then $h \in$ R (β) on S and we have

$$\int_a^b f d\alpha = \int_c^d h\, d\beta$$

(i.e.) $\int_{g(c)}^{g(d)} f(t) d\alpha\,(t) = \int_c^d f[g(x)] d\{\alpha[g(x)]\}$

Proof:

Let f and α be real valued function defined on [a,b]

Let p = { a = $x_0, x_1, \ldots\ldots, x_n$= b }$\in \wp$ [a, b]

Let $t_k \in [\, x_{k-1}, x_k]$ $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\epsilon > 0$ be given

Let g be a strictly monotonic function on [c, d]

Assume that $a = g(c)\ and\ b = g(d)$

Let h and β be the composite functions $\ni$

$h(x) = f[g(x)], \beta(x) = \alpha[g(x)]\ if\ x \epsilon [c, d]$ ……. (1)

To prove: $h \in$ R (β) on $[c, d]$ and $\int_a^b f d\alpha = \int_c^d h\, d\beta$

Now, $f \in$ R (α)

$\Rightarrow$ *their exist* $A \in$ R $\ni \forall\, \epsilon > 0$ their exist P_ε of [a, b] such that

$\forall$ p finer than P_ε *and* $t_k \in [x_{k-1}, x_k]$, we have

$|S(P, f, \alpha) - A| < \varepsilon$ and $A = \int_a^b f d\alpha$ ....(2)

Also assume that

g is strictly monotonic increasing and continuous on [c, d]

$\Rightarrow g$ is 1-1 and onto from [c, d] and [a, b]

and g^{-1} exists and g^{-1} is also strictly increasing and continuous on [a,b]

$\therefore \forall$ partition $p'\{c = y_0, y_1, \ldots\ldots, y_n = d\}$ of [c, d]

Their exists one and only partition p = { a = $x_0, x_1, \ldots\ldots, x_n$= b }of [a,b] with $x_k \in g(x_k)$

(i.e.) $p = g(p')$

$\Rightarrow p' = g^{-1}(p)$

Let $p'_\varepsilon = g^{-1}(P_\varepsilon)$ be the corresponding partition of [c, d]

Let $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \alpha(\beta_k) - \alpha(\beta_{k-1})$

also $t_k = g(u_k) and\ x_k = g(y_k)$ ....(3)

Now, $\forall$ partition p' finer than p'_ε , we have

$|S(p', h, \beta) - A| = |\sum_{k=1}^{n} h(u_k)\Delta\beta_k - A|$

$= |\sum_{k=1}^{n} h(u_k)\ [\beta(y_k) - \beta(y_{k-1})] - A\ |$

$= |\sum_{k=1}^{n} \mathrm{f}(\mathrm{g}(u_k))[\alpha(g(y_k)) - \alpha(g(y_{k-1})] - A\ |$

$\leq |\sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(x_{k-1})] - A|$

$\leq |\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A|$

$= |S(P, f, \alpha) - A|$

$< \varepsilon$ (by equation (2))

$|S(p', h, \beta) - A| < \varepsilon$

$\therefore h \in$ R (β) and $\int_c^d h\, d\beta = A$

$$\Rightarrow \int_c^d h\, d\beta = \int_a^b f d\alpha$$

(i.e.) $\int_a^b f d\alpha = \int_c^d h\, d\beta \Rightarrow \int_a^b f(t) d\alpha\,(t) = \int_c^d h(x) d\beta\,(x)$

$$\Rightarrow \int_{g(c)}^{g(d)} f(t) d\alpha\,(t) = \int_c^d f[g(x)] d\{\alpha[g(x)]\}$$

Note:

When $\alpha(x) = x$, the above Theorem applies to Riemann integrals.

Reduction to a Riemann Integral

Theorem 2.11:

Assume f $\in$ R (α) on [a,b] and assume that α has a continuous derivative α' on [a,b]. Then the Riemann integral $\int_a^b f(x)\, \alpha'(x) dx$ exists and we have

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha'(x) dx.$$

Proof:

Let f & d be real-valued functions defined on [a,b]

Let p = { a = $x_0, x_1, \ldots\ldots, x_n$= b }$\in$ P[a,b] Let $t_k \in [\, x_{k-1}, x_k]$

$\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\epsilon > 0$ be given

Given f $\in$ R (α) and α has a continuous derivative α' on [a,b].

To Prove: $\int_a^b f(x)\,\alpha'(x)dx$ exists and $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,\alpha'(x)dx$.

Now, f ∈ R (α) => $\exists\, A \in R \ni \forall \epsilon_1 = \epsilon_2 > 0 \;\exists\, P_\epsilon$ of [a,b] ∋

$\forall\, P$ finer than P_{ϵ_1} and $t_k \in [\, x_{k-1}, x_k]$,we have

| S(P, f α) – A | < ϵ *and* A=$\int_a^b f\, d\alpha$ ……….(1)

Also, Given α ' exists and is continuous on $[a, b]$

By-Mean-Value Theorem, $\exists\, v_k \in [x_{k-1}, x_k]$

$$\alpha(x_k) - \alpha'(x_{k-1}) = \alpha'(v_k)\,(x_k - x_{k-1})$$
$$\Delta\alpha_k = \alpha'(v_k).\Delta x_k$$

$\Rightarrow \Delta\alpha_k = \alpha'(v_k) \cdot \Delta x_k$ ………… (3)

α' is continuous on $[a, b]$

$\Rightarrow$ α' is unisormly_ontinuous on $[0, b]$

$\Rightarrow$ given $\varepsilon_2 > 0\, \exists, \delta > 0, \ni$

$$|x - y| < \delta \Rightarrow |\alpha'(x) - \alpha'(y)| < \varepsilon_2 = \frac{\varepsilon}{2\mathrm{M}(b - \mathrm{a})}$$

If we take a partition P_{ε_2} with norm $\|P_{\varepsilon_2}\| < \delta$,

& partition P finer than P_{ε_2}, we have

$|\alpha'(t_k) - \alpha'(v_k)| < \varepsilon_2 = \frac{\varepsilon}{2M(b-a)}$ ………… (4)

Let $P_\varepsilon = P\varepsilon_1 \cup P_{\varepsilon_2}$

Then $\forall\, P$ finer than P_ε we have

$|S(P, f\alpha') - A| = |S(P, f\alpha') - S(P, f, \alpha) + S(P, f, \alpha) - A|$

$\leqslant |S(P, f\alpha') - S(P, f, A)| + |S(P, f, \alpha) - A|$

$= |\sum_{k=1}^{n}(f\alpha')(t_k)\cdot\Delta x_k - \sum_{k=1}^{n}f(t_k)\cdot\Delta\alpha_k|+,|S(\mathrm{P},\mathrm{f},\alpha)-A|$

$= |\sum_{k=1}^{n}f(t_k)\alpha'(t_k)\cdot\Delta x_k - \sum_{k=1}^{n}f(t_k)\alpha'(v_k)\Delta x_k| + |S(\mathrm{P},\mathrm{f},\alpha)-\mathrm{A}|$

(by equation (3))

$= |\sum_{k=1}^{n}f(t_k)[\alpha'(t_k)-\alpha'(v_k)]\Delta x_k| + |\, S(\mathrm{P},\mathrm{f},\alpha - A)\,|$

$< \ |\sum_{k=1}^{n}M\frac{\varepsilon}{2(b-a)}.\Delta x_k| \ +\frac{\varepsilon}{2}$ (by equation 1,2 and 4)

$= \frac{\varepsilon}{2(b-a)}|\sum_{k=1}^{n}\Delta x_k| + \frac{\varepsilon}{2}$

$= \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2}$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

(i)$|\, S(P, f\alpha') - A\,| < \varepsilon.$

$\Rightarrow \int_a^b f(x)\alpha'(x)dx$ exists and

$\int_{\mathrm{a}}^{b} f(x)\cdot\alpha'(x)dx = A$

(i.e.), $\int_a^a f(x)\alpha'(x)dx = \int_a^b f d\alpha$

(i.e.), $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx,$

Step Functions as Integrators

Note:

- If ' α ' is constant on $[a,b]$, then the Integral $\int_a^b f d\alpha$ exists and has value 0 (i.e.) $\int_a^b f d\alpha = 0$ ($\because s(P,f,\alpha) = 0$)
- If ' α ' is constant except for a jump discontinuity at one point, then the integral $\int_a^b f d\alpha$ need not exist;

- If $\int_a^b f d\alpha$ does exist, its value need not be zero.

Theorem 2.12:

Given a $a < cb$. Define α on $[a, b]$ as follows:

The values $\alpha(a), \alpha(c), \alpha(b)$ are arbitrary;

$\alpha(x) = \alpha(a)$ if $a \leq x < c$ and $\alpha(x) = \alpha(b)$ if $c \leq x < b$

Let $'f'$ be defined on [a, b] in such a way that at least one of the functions 'f ' or ' α ' is continuous from the left at ' c ' and at least one is continuous from the fight at ' c '. Then $f \in R(\alpha)$ on [a b], and we have

$$\int f d\alpha = f(0)\ [\alpha(+) - \alpha(-)]$$

Proof:

Given $a < c < b$

Let f & α be real-valued functions defined on $[a, b]$ Define ' α ' on $[a, b]$ as follows:

$$\alpha(x) = \begin{cases} \alpha(a) & \text{if } a \leq x < c \\ \alpha(c) & \text{if } x = c \\ \alpha(b) & \text{if } c < x \leq b \end{cases} \quad \ldots\ldots\ldots (1)$$

where $\alpha(a), \alpha(c), \alpha(b)$ are arbitrary.

Let $P = \{a = x_0, x_1, \dots, x_{k-1}, x_k, \dots; x_n^b\} \in P[a, b]$

Let $t_k \in [x_{k-1}, x_k]$ & $\Delta d_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $c \in [t_{k-1}, t_k]$

Let $\varepsilon > 0$ be given

Given atleast one of the functions ' f ' or ' α ' is continuous from the left at ' c ' and at least one is continuous from the right at ' C '.

To prove: $f \in R(\alpha)$ & $\int_a^{fd\alpha} = f(c)[\alpha(c+) - \alpha(c-1)]$

Consider the corresponding Riemann-Stieltjes sum with respect to p :

$$= s(P_2, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$

$$\Rightarrow s(p_2, f, \alpha) = \sum_{k=1}^{n} f(t_k)[\alpha(x_k) = \alpha(x_{k-1})]$$

$$= f(t_1)[\alpha(x_1) = \alpha(x_0)] + f(t_2)[\alpha(x_2) - \alpha(x_1)] + \cdots + f(t_{k-1})$$
$$[\alpha(x_{k-1}) = \alpha(x_{k-2})] + f(t_k)[\alpha(x_k) - \alpha(x_k) + \cdots + f(t_n)[\alpha(x_n) - \alpha(x_{n-1})]$$

$$= f(t)[\alpha(a) - \alpha(a)] + f(t_2)[\alpha(a) - \alpha(a) + \cdots + f(t_{k-1})[\alpha(c) - \alpha(a)] + f(t_k)$$
$$[\alpha(b) = \alpha(c)] + \cdots + f(t_n)[\alpha(b) - a(b)]$$

$$\therefore S(R, f, \alpha) = f(t_{k-1})[\alpha(c) - \alpha(c-1)] + f(t_k)[\alpha(c+) - \alpha(c)]$$

Let $\Delta = S(p, f, \alpha) - f(C) \cdot [\alpha(C+) - \alpha(C-)]$

Now,

$$|\Delta| = |s(p, f, \alpha) - f(c) \cdot [\alpha(c+) - \alpha(c-)]$$
$$= |f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)] - f(c)[\alpha(c) - \alpha(c-)]$$
$$-f(c)[\alpha(c+) - \alpha(c-)] \mid$$
$$= |[f(t_{k-1}) - f(c)] \cdot [\alpha(c) - \alpha(c\,-)] + [f(t_k) - f(c)][\alpha(c\,+) - \alpha(c)]|$$

$$\therefore |\Delta| \leqslant |f(t_{k-1}) - f(\mathrm{c})| \cdot |\alpha(c) - \alpha(c-)| + |f(t_k) - f(\mathrm{c}) \mid |\alpha(c\,+) - \alpha(c)| \ldots (2)$$

Case (i) ' f ' is continuous on both sides at ' c '.

$$\therefore Given\ \varepsilon > 0, \exists\ \delta > 0 \Rightarrow:$$
$$\| P \| < \delta \Rightarrow |f(t_{k-1}) - f(c)| < \varepsilon \text{ and} |f(t_k) - f(c) \,| < \varepsilon$$
$$: |\Delta| \leqslant \varepsilon \cdot |\alpha(c) - \alpha(c-)| + \varepsilon \cdot |\alpha(c+) - \alpha(c)|$$

This inequality holds whether or not ' f ' is continuous at ' c '.

case (ii) ' r ' is discontinuous on both sides at ' c '

$\Rightarrow \alpha$ ' is continuous at 'e on both sides at ' C '

$\Rightarrow \alpha'(c) = \alpha(C) = \alpha(Ct)$

We get, $|\Delta| = 0$

Case(iii) ' f ' is continuous from left & ' f ' is discontinuous from the right it $c \Rightarrow$ ' α ' is continuous from the right at ' c '

$\Rightarrow \alpha(c) = \alpha(ct)$

∴ we get, $|\Delta| \leqslant \varepsilon|\alpha(c) - \alpha(c-)|$

Case (iv) ' f ' is continuous from the right of

' f ' is discontinuous from the left at ' C '

$\Rightarrow$ ' α ' is continuous from the left at ' C '

$\Rightarrow \alpha(0) = \alpha(c-)$

∴ we get, $|\Delta| \leqslant \varepsilon \cdot |\alpha(c+) - \alpha(c)|$

From the above four cases; we get $f \in R(\alpha)$

& $\int_a^b f d\alpha = f(0)[\alpha(c+) - \alpha(c-)]$

Note:

The value of a Riemann - stieltjes integral can be altered by changing the value of ' f ' at a single point.

Example 2.13:

This example shows that the existence of the integral k can also be affected by a change of the value of ' f ' at a single point.

Let $\alpha(x) = \begin{cases} 0, & if\ x \neq 0 \\ -1, & if\ x = 0 \end{cases}$

f(x) = 1 if -1≤ x ≤1

by Theorem 2.12 we get $\int_{-1}^{1} f d\alpha = f(0)[\alpha(0+) - \alpha(0-)$

= 1. [0-0] = 0

$\int_{-1}^{1} f d\alpha = 0$, If we re-define 'f' so that

$f(x) = \begin{cases} 1, & if\ x \neq 0 \\ 2, & if\ x = 0 \end{cases}$

Then

S(p, f, α) = $f(t_{k-1})[\alpha(0) - \alpha(x_{k-1})] + f(t_k)[\alpha(x_k) - \alpha(0)]$

= $f(t_{k-1})[-1 - 0] + f(t_k)[0 - (-1)]$

S(p, f, α) = $f(t_{k-1}) + f(t_k)$

Where $x_{k-2} \leq t_{k-1} \leq 0 \leq t_k \leq x_k$

The value of this sum is 0,1 (or) -1, depending on $\int_{-1}^{1} f d\alpha$ does not exist.

Note:

In the Riemann integral $\int_a^b f(x)dx$; the values of 'f' can be changed at a finite numbers of points without affecting either the existence or the value of the integral.

To prove this, consider f(x) = $\begin{cases} 0\ \ if\ \ x \in [a,b]\backslash\{c\} \\ c\ \ if\ x = c \end{cases}$

S(P, f) ≤ | f(c)| . || P||

||p|| can be made arbitrarily small,

$\int_a^b f(x)dx = 0$

Reduction of a Riemann-Stieltjies integral to a finite sum:

Definition 2.14:

Let ' f ' be defined on a closed interval $[a, b]$. If ($f(c) - f(c-)$) is exist at some interior point ' C ' then

(a) If $f(c) - f(c-)$ is called the left hand jump of ' f' at ' c '.

(b) $f(c+) - f(c)$ is called the right hand jump of ' f ' at ' C '.

(c) $f(c) - f(c)$ is called the Jump of ' f ' at ' c '. If any one of the these three numbers is different from ' 0 '. then ' c ' is called a jump discontinuity of "f."

Definition 2.15: [Step Function]

A function ' α ' defined in $[a, b]$. is called a Step function if there exists a partition $a = x_1 < x_2 < \cdots < x_n = b$:

' α ' is constant on each open sub interval (x_{k-1}, x_k).

Note: Jump at $x_k = \alpha_k = \alpha(x_k +) - \alpha(x_k -), 1 < k < n$

- Jump at $x_1 = \alpha_1 = \alpha(x_1 +) - \alpha(x_1)$
- Jump at $x_n = \alpha_0 = \alpha(x_n) - \alpha(x_n -)$

Example: $\alpha(x) = \begin{cases} -0.5, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ 1.5, & 1 < x < 2 \\ 2.5, & 2 < x < 3 \end{cases}$

Theorem 2.16: [Reduction of a Riemann-Stieltjes Integral to a finite sum]

Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $x_1, x_2, \dots, x_n$ are as described in Definition 2.15. Let ' f ' be defined on $[a, b]$ in such a way that not both ' f ' and ' α ' are discontinuous from the right or from the left at each x_k then $\int_a^b f d\alpha$ exists and we have $\int_a^b f(x) d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k$.

Proof:

Let ' f be a real -valued function on $[a, b]$

let ' α ' be a step function on $[a, b]$

$\Rightarrow \exists$ a partition $P=\{a=x_1, x_2, \ldots, x_n=b\}$

where $a=x_1<x_2<\cdots<x_n=b$;

Their exist ' α ' is constant on each (x_{k-1}, x_k).

Let the Jump at x_k be $\alpha_k=\alpha(x_k+)-\alpha(x_k-)$ ... ...(1)

To prove: $\int_a^b f d\alpha$ exists and $\int_a^b f d\alpha=\sum_{k=1}^n f(x_k) \cdot \alpha_k$

Let $t_k \leq x_k \leq t_{k+1}$

Given not both ' f ' & ' α ' are discontinuous from right or from left at each ' x_k'

By Theorem2.7,

$$\int_a^b f d\alpha=\int_{t_1}^{t_2} f d\alpha+\int_{t_2}^{t_3} f d\alpha+\cdots+\int_{t_k}^{t_{k+1}} f d\alpha+\cdots+\int_{t_0}^{t_{n+1}} f d\alpha$$

$$\Rightarrow \int_a^b f d\alpha=\sum_{k=1}^n \int_{t_k}^{t_{k+1}} f d\alpha \ldots \ldots \ldots (2)$$

Also by Theorem 2.12,

$$\int_{t_k}^{t_{k+1}} f d\alpha=f(x_k)[\alpha(x_k t)-\alpha(x_k-2)],$$

where $t_k \leqslant x_k \leqslant t_{k+1}, k=1,2, \ldots, n$

$$\therefore (1) \Rightarrow \int_a^b f d\alpha=\sum_{k=1}^n f(x_k)[\alpha(x_k+)-\alpha(x_k-)]$$

$$\Rightarrow \int_a^b f d\alpha=\sum_{k=1}^n f(x_k)\alpha_k\ [by 1]$$

Definition 2.17: (Greatest Integer Function)

- one of the simplest step functions is the greatest-integer function
- Its value at 'x' is the greatest integer which is less than or equal to 'x' and is denoted by $[x]$
- Thus [x] is the unique integer satisfying the inequalities

  $[x] \leq x < [x] + 1$
- The graph of the greatest integer function is gn below:
- For ex, $[2 \cdot 4] = 2; [\pi] = 3; [-4 \cdot 2] = -5.$

Theorem 2.18:

Every finite sum of real numbers can be written as a Riemann-Stieltjes. integral. In fact, given a sum $\sum_{k=1}^{n} a_k$, define ' f ' on $[0, \mathrm{n}]$ as follows:

$f(x) = a_k$ if $k - 1 < x \leqslant k$ $(k = 1,2, \dots, n); f(0) = 0$

Then $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x) d[x]$,

where $[x]$ is the greatest integer $\leq x$.

Proof:

Let the finite sum of real numbers be $\sum_{k=1}^{n} a_k$

Define $f: [0, \mathrm{n}] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x = 0 \\ a_k, & k - 1 < x \leq k, k = 1,2, \dots, n. \end{cases}$

(i.e.), $f(x) = \begin{cases} 0 & , x = 0 \\ a_1, & 0 < x \leq 1 \\ \dot{a}_{n-1} & n - n < x \leqslant n - 1 \\ a_n, & n - 1 < x \leqslant n \end{cases}$

Define $\alpha: [0, \mathrm{n}] \to \mathbb{R}$ by $\alpha(x) = [x]$, where.

$[x]$ is the greatest integer $\leqslant x$

$$\text{(i.e.)}, \alpha(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ \vdots & \\ \vdots & \\ n-2 & n-2 \le x < n-1 \\ n-1 & n-1 \le x < n \end{cases}$$

To Prove: $\int_0^n f(x) d[x] = \sum_{k=1}^n a_k$ x = n let partition of [0,n] = {0,1,2,….,n} and jump at k, $\alpha_k = \alpha(k+) - \alpha(k-)$

From equation (1) and (2) we get,

'f ' is continuous from the left at each integer $k = 1,2, \dots, n$ and ' α ' is continuous from the right and having jump ' I ' at each integer $k = 1,2 \dots, n$.

By Theorem 2.16, $\int_0^n f d\alpha$ exists

and $\int_0^n f(x) d\alpha(x) = \sum_{k=1}^n f(x_k)\, \alpha_k$

$$\Rightarrow \int_0^n f(x) d[x] = \sum_{k=1}^n f(k)(\alpha(k+) - \alpha(k-))$$

$$= \sum_{k=1}^{n} a_k([k+] - [k-])$$

$$= \sum_{k=1}^{n} a_k\big((\text{k}) = (k-1)\big)$$

$$\int_0^n f(x) d[x] = \sum_{k=1}^n a_k$$

Euler's Summation Formula

Note:

Euler's summation formula relates the integral of a function over an interval $[a, b]$ with the sum of the function values at the integers in $[a, b]$

It also used to approximate integrals by sums or conversely, to estimate the values of certain sums by means of integrals.

Theorem 2.19: [Euler's Summation Formula]

We have has a continuous derivative f' on $[a, b]$, then

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b)),$$

where $((x)) = x - [x]$. When $a + b$ are integers, this becomes

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx + \frac{f(a) + f(b)}{2}$$

Proof:

Given, ' f ' has a continuous derivative f' on $[a, b]$

By Theorem 2.10, the integration by parts of $R - S$ integral, we get,

$$\int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

Replace $\alpha(x), \alpha(a), \alpha(b)$ by $x - [x], a - [a], b - [b]$ respectively, we get,

$$\int_a^b f(x)d(x - [x]) + \int_a^b (x - [x])df(x) = f(b)(b - [b]) - f(a)(a - [a])$$

$$\Rightarrow \int_a^b f(x)dx - \int_a^b f(x)d[x] + \int_a^b ((x))df(x) = f(b)((b)) - f(a)((a))$$

$$\Rightarrow \int_a^a f(x)dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b))\Big) = \int_a^b f(x)d[x]$$

By Theorem 2.18, we get, $\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)d[x]$

$$\therefore \sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b)) \ldots\ldots. (1)$$

When a and b are integers,

$((x)) = x - [x]; \ ((a)) = a - [a] = a - a = 0$

$\big((b)\big) = b - [b] = b - b = 0$

$\therefore (1) \Rightarrow \sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx$

$\Rightarrow \sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)(x-[x])dx - \frac{1}{2}\int f'(x)dx$

$\Rightarrow \sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x) - \left(x-[x]-\frac{1}{2}\right)dx + \frac{1}{2}\int_a^a f'(x)dx$

$\Rightarrow \sum_{a<n\leqslant b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(x-[x]-\frac{1}{2}\right)dx + [\frac{1}{2}f(x)]_a^b$

$\sum_{a<n\leqslant b} f(n) - \int_a^b f(x)dx + \int_a^b f'(x)\left(x-[x]-\frac{1}{2}\right)dx + \frac{1}{2}[f(b)-f(a)]$

$\sum_{a<n\leqslant b} f(n) + f(a) = \int_a^b f(x)dx + \int_a f'(x)\left(x-[x]-\frac{1}{2}\right)dx + \frac{1}{2}f(b) - \frac{1}{2}f(a) + f(a)$

$\Rightarrow \sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(x-[x]-\frac{1}{2}\right)dx + \frac{1}{2}(f(a)+f(b)$

Monotonically Increasing Integrators. Upper and Lower integrals

Note:

- When ' α ' is increasing, the differences $\Delta\alpha_k = \alpha(x_k) - \alpha(x_k)$ which appear in the Riemann-Stieltjes sums are all non-negative. (i.e.), $\Delta\alpha k \geqslant 0$ when α is increasing.
- $\alpha \nearrow$ on $[a,b]$ to means that α is increasing on $[a,b]$.

Definition 2.20:

To find the area of the region under the graph of a function ‘ f ‘ we consider Riemann sums $S(P,f) = \sum f(t_k)\Delta x_k$ as approximation to the area by means of rectangles. Let P be a partition of [a, b].

Then the upper and lower Riemann sums of a function ‘f ’ are

$$U(P,f) = \sum M_k(f)\Delta x_k$$

and $L(P,f) = \sum M_k(f)\Delta x_k$

where, $M_k(f) = \text{Sup}\,\{f(x): x \in [x_{k-1}, x_k]\}$

$$m_k(f) = \inf\{f(x): x \in [x_{k-1}, x_k]\}$$

Our geometric intuition tells us that the upper sums are at least as big as the area we seek, whereas the lower sums cannot exceed this area.

Then the upper integral of ‘f is the inf of all upper sums and the lower integral of ‘ f ‘ is the sup of all lower sums.

(i.e.) Upper integral of ‘ f ‘ $= \overline{\int_a^b} f\,dx = \inf\{U(P,f) : P \in \wp[a,b]\}$

Lower integral of ‘ f ‘ $= \underline{\int_a^b} f\,dx = \text{Sup}\{L(P,f) : P \in \wp[a,b]\}$

If ‘ f ‘ is a continuous function, then $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f\,dx$

Definition 2.21:

Let P be a partition of [a, b] and let $M_k(f) = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]$

$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

The numbers $U(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta x_k$

and $L(P,f,\alpha) = \sum_{k=1}^{n} m_k(f)\Delta x_k$ are called respectively, the upper and lower stieltjes sums of ‘ f ‘ with respect to ‘ α ‘ for the partition ‘ P ‘.

Note:

α increasing on [a, b] $\Rightarrow L(P,f,\alpha) \leq S(P,f,\alpha) \leq U(P,f,\alpha)$

Let $t_k \in [x_{k-1}, x_k]$

Then clearly, $\inf\{f(x)\} \leq f(t_k) \leq \text{Sup}\{f(x)\}, \quad x \in [x_{k-1}, x_k]$

(i.e.), $m_k(f) \leq f(t_k) \leq M_k(f)$ ….(1)

If α increasing on [a, b], then $\Delta\alpha_k \geq 0$

$\therefore (1) \Rightarrow m_k(f)\Delta\alpha_k \leq f(t_k)\Delta\alpha_k \leq M_k(f)\Delta\alpha_k$

$\Rightarrow \sum m_k(f)\Delta\alpha_k \leq \sum f(t_k)\Delta\alpha_k \leq \sum M_k(f)\Delta\alpha_k$

(i.e.), $L(P,f,\alpha) \leq S(P,f,\alpha) \leq U(P,f,\alpha)$

Hence if α increasing on [a, b] then, $L(P,f,\alpha) \leq S(P,f,\alpha) \leq U(P,f,\alpha)$

Theorem 2.22:

Assume that α increasing on $[a, b]$. Then:

(i)If P' is finer than P, we have $U(P', f, \alpha) \leqslant U(P, f, \alpha)$ & $L(P', f, \alpha) \geqslant L(P, f, \alpha)$

(ii) For any two partitions P_1 and P_2, we have $L(P_1, f, \alpha) \leqslant U(P_2, f, \alpha)$.

Proof:

Assume that α increasing on $[a, b]$

(i) Let $P, P' \in \mathbb{P}[a, b]$

Given P' is finer thath P (i.e.), $P \subset P'$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

It suffices to prove, when P' contains exactly one more point than P, say the point ' C '.

let $P' = \{a = x_0, x_1, \dots, x_{i-1}, c, x_i, \dots, x_n = b\}$

Then the upper stieltjes sums of ' f ' ' w.r.to ' α ' for P is

$U(P, f, \alpha) = \sum_{k=1}^{n} M_k \cdot (f) \Delta\alpha_k$

where $M_k(f) = \sup\{f(x): x \in [x_{k-1}, x_k]\}$

consider the upper stieltjes sum of ' f ' w.r.to ' α ' for P ' is

$U(P', f, \alpha) = \sum_{\substack{K=1 \\ K \neq 1}}^{n} M_k(f) \Delta\alpha_k + M'(f)[\alpha(c) - \alpha(x_{i-1})] + M''(f)[\alpha(x_i) - \alpha(c)]$

where $M'(f) = \sup\{f(x): x \in [x_{i-1}, c]\}$

$M''(f) = \sup\{f(x): x \in [c, x_i]\}$

clearly, $M'(f) \leqslant M_i(f) + M''(f) \leqslant M_i(f)$ …… (1)

Now, $M'(f)[\alpha(c) - \alpha(x_{i-1})] + M''(f)[\alpha(x_i) - \alpha(c)] \leqslant M_i(f)[\alpha(c) - \alpha(x_{i-1})] + M_i(f)[\alpha(x_i) - \alpha(c)]$ (by equation (1))

$= \mathrm{M_i(f)[\alpha(c) - \alpha(x_{i-1}) + \alpha(x_i) - \alpha(c)]}$

$= \mathrm{M_i(f)[\alpha(x_i) - \alpha(x_{i-1})]}$

$\sum_{k=1}^{n} M_k(f)\Delta\alpha_k + M'(f)[\alpha(c) - \alpha(x_{i-1})] + M''(f)[\alpha(x_i) - \alpha(c)]$

$\leq \sum_{k=1}^{n} M_k(f)\Delta\alpha_k + M_i(f)[\alpha(x_i) - \alpha(x_{i-1})] \leq \sum_{k=1}^{n} M_k(f)\Delta\alpha_k$

$\therefore (U(P', f, \alpha) \leq U(P, f, \alpha)$

$L(P', f, \alpha) \geq L(P, f, \alpha)$

(ii) Let P_1 & P_2 be two partitions of $[a, b]$.

Let $P = P_1 \cup P_2$

$\Rightarrow P_1 \subseteq P$ &$P_2 \subseteq P$

Then by (i), $\mathrm{L}(P_1, f, \alpha) \leqslant \mathrm{L}(P, f, \alpha) \leqslant U(P, f, \alpha) \leqslant U(P_2, f, \alpha)$

$\Rightarrow \; L(P_1, f, \alpha) \leqslant U(P_2, f, \alpha)$

Note:

If $\alpha \nearrow$ on $[a, b]$, then $m[\alpha(b) - \alpha(a)] \leq M \cdot [\alpha(b) - \alpha(a)]$

Where

$M = \sup\{f(x): x \in [a, b]\}$ $m = \inf\{f(x): x \in [a, b]\}$

For, $m \cdot [\alpha(b) - \alpha(a)] \leq L(P_1, f, \alpha)$

$\leq u(P_2, f, \alpha)$

$\leq M \cdot [\alpha(b) - \alpha(a)]$

$\therefore m[\alpha(b) - \alpha(a)] \leq M \cdot [\alpha(b) - \alpha(a)]$

Definition 2.23:

Assume that $\alpha \nearrow$ on $[a, b]$.The Upper Riemann-stieltjes Integral of $'f'$ w.r.t ' α ' is defined as follows: $\bar{I}(f, \alpha) = \int_a^{-b} f d\alpha = inf\{U(P, f, \alpha); p \in \mathcal{P}[a, b]\}$

The Lower Riemann- stieltjes Integral of ' f ' w.r.t ' α ' is defined as follows: $\underline{I}(f, \alpha) = \int_{-a}^{b} f d\alpha = \sup\{L(P, f, \alpha): p \in \mathcal{P}[a, b]\}$

If $\int_{-a}^{b} f d\alpha = \int_a^{-b} f d\alpha$, then ' f ' is said to be Riemann- stieltjes integrable on $[a, b]$.

Note:

When $\alpha(x) = x$, then $U(P, f)\&L(P, f)$ are called the upper and lower Riemann sums. The corresponding Upper Riemann Integral is $\int_a^{-b} f(x)dx = \inf\{U(P, f); P \in \mathcal{P}[a, b]\}$

The Lower Riemann integral is $\int_{-a}^{b} f(x)dx = \sup\{L(P, f): P \in \mathcal{P}[a, b]\}$

If $\int_{-a}^{b} f(x)dx = \int_a^{-b} f(x)dx$, then ' f ' is said to be Riemann Integrable on $[a, b]$.

Theorem 2.24:

Assume that $\alpha \nearrow$ on $[a, b]$. Then $\underline{I}(f, \alpha) \leq \bar{I}(f, \alpha)$ (i.e.) $\int_{-a}^{b} f d\alpha \leq \int_a^{b} f d\alpha$

Proof:

Let P be a partition of $[a, b]$

Let $\varepsilon > 0$ be given

Then there exsist a partition $P_1 \ni: U(P_1, f, \alpha) < \bar{I}(f, \alpha) + \varepsilon$

Let P be finer than P_1

(i.e.),$P_1 \subseteq P$

Then by Theorem 2.22,

$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$ & $U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$

$\Rightarrow L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

$\Rightarrow L(P_1, f, \alpha) \leq U(P_1, f, \alpha)$

$\Rightarrow L(P, f, \alpha) \leq \bar{I}(f, \alpha) + \varepsilon$ ……… (2)

(i.e.)., $\bar{I}(f, \alpha) + \varepsilon$ is an upper bound to all lower sums $L(P, f, \alpha)$

From the definition of supremum and by (2), we get,

$\sup\{L(p, f, \alpha): P \in \mathcal{P}[a, b]\} \leq \bar{I}(f, \alpha) + \varepsilon$

(i.e.) $\underline{I}(f, \alpha) \leq \bar{I}(f, \alpha) + \varepsilon$

$\because \varepsilon > 0$ is arbitrary, we get, $\underline{I}(f, \alpha) < \bar{I}(f, \alpha)$

Example 2.25:

It is easy to give an example in which $\underline{I}(f, \alpha): < \bar{I}(f, \alpha)$.

Let $\alpha(x) = x$

Define ' f ' on [0,1] as follows.

$$f(x) = \begin{cases} 1 \; if \; x \; is \; rational \\ 0 \; if \; x \; is \; irrational \end{cases}$$

Then for every partition P of [0,1], we have

$M_k(f) = \sup\{f(x): x \in [x_{k-1}, x_k]\} = 1$

and $m_k(r) = \inf\{f(x): \text{x} \in [x_{k-1}, x_k]\} \equiv 0$

(i.e.) $M_k(f) = 1$ & $m_k(f) = 0$

$\because$ Every subinterval contains both rational and irrational numbers

$\therefore U(P, f) = \sum M_k(f) \cdot \Delta x_k = \sum 1 \cdot \Delta x_k = \sum x_k = b = a = 1$

$L(P,f) = \sum\ m_k(f) \cdot \Delta x_k = \sum\ 0 \cdot \Delta x_k = 0$

(i.e.)., $U(P,f) = 1$ & $L(P,f) = 0\ \forall P$.

$\therefore$ For $[a,b] = [0,1]$,$\int_a^{-b} f(x)dx = inf\{\mathrm{U}(P,f): P \in \mathcal{P}[a,b]\} = 1$

$and \int_{-a}^{b} f(x)dx = \sup\{\mathrm{L}(P,f): P \in \mathcal{P}[a,b]\} = 0$

The same result holds if $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Additive and Linearity Properties of Upper and Lower Integrals.

Theorem 2.26:

Let ' f ' be a function defined on $[a,b]$ and let ' α ' be an increasing function on $[a,b]$. Then for $any\ c \in (a,b)$ we have that

a) $\int_a^{-b} f d\alpha = \int_a^{-c} f d\alpha + \int_c^{-b} f d\alpha$ b) $\int_{-a}^{b} f d\alpha = \int_{-a}^{c} f d\alpha + \int_{-c}^{b} f d\alpha$

Proof:

Let ' f ' be a function on $[a,b]$

Let ' α' be an increasing function on [a,b]

Let $c \in (a,b)$.

Let $P = \{a = x_0, x_1, \dots, x_n = c = y_0, y_1, \dots, y_m = b\}$ be a partition of $[a,b]$

Let $P_1 \equiv \{a = x_0, x_1, \dots, x_n = c\} - d, P_2 = \{c = -y_0, y_1, \dots, y_m = b\}, be$ the partitions of $[a,c]$ & $[c,b]$. respectively.

a) Now, $\int_a^{b} f d\alpha = \inf\{U(P,f,\alpha): P \in \mathcal{P}[\Omega,b]\}$

$$= \inf\left\{\sum_{k=1}^{n} \sup\{f(x): x \in [x_{k-1}, x_k]\}\Delta\alpha_k + \sum_{k=1}^{m} \sup\{f(y): y \in [y_{k-1}, y_k]\}\Delta\alpha_k\right\}$$

$$= \inf\left\{\sum_{k=1}^{n} \sup\{f(x): x \in [x_{k-1}, x_k]\}\Delta\alpha_k: p_l \in \mathcal{P}[a, c]\right\}$$

$$+ \inf\left\{\sum_{k=1}^{m} \sup\{f(y): y \in [y_{k-1}, y_k]\}\Delta\alpha_k: P_2 \in \mathcal{P}[c, b]\right\}$$

$$= \inf\{U(P_1, f, \alpha): P_1 \in \mathcal{P}[a, c]\} + \inf\{U(P_2, f, \alpha): P_2 \in \mathcal{P}[c, b]\} = \int_{\mathrm{a}}^{-c} f d\alpha + \int_{c}^{-b} f d\alpha$$

$$\therefore \int_{a}^{-b} f d\alpha = \int_{a}^{-c} f d\alpha + \int_{c}^{-b} f d\alpha$$

Similarly, we can prove that

$$\int_{-a}^{b} f d\alpha = \int_{-a}^{c} f d\alpha + \int_{-c}^{b} f d\alpha.$$

Theorem 2.27:

Let ' f ' and ' g ' be any functions defined on $[a, b]$ and let ' α ' be an increasing function on $[a, b]$. Then

a) $\int_{a}^{-b} (f + g) d\alpha \leq \int_{0}^{-b} f d\alpha + \int_{a}^{-b} g d\alpha$

b) $\int_{-a}^{b} (f + g) d\alpha \geq \int_{-a}^{b} f d\alpha + \int_{-a}^{b} g d\alpha$

Proof:

Let ' f ' and ' g ' be any functions defined on $[a, b]$

let ' α ' be an increasing function on $[a, b]$

Let $P = \{a = x_0, x_1, \dots, x_n = b\} \in \mathcal{P}[a, b]$

Clearly, $f(x) \leq \sup\{f(x): x \in [x_{k-1}, x_k]\}$

$g(x) \leq \sup\{g(x): x \in [x_{k-1}, x_k]\}$

$\Rightarrow f(x) + g(x) \leq \sup\{f(x): x \in [x_{k-1}, x_k]\} + \sup\{g(x): x \in [x_{k-1}, x_k]\}$

$\Rightarrow \sup\{f(x)+g(x): x \in [x_{k-1}, x_k]\}$

$\leq \sup\{f(x): x \in [x_{k-1}, x_k]\} + \sup\{g(x): x \in [x_{k-1}, x_k]\}$

(i.e.) $M_k(f+g) \leq M_k(f) + M_k(g)$

$$\Rightarrow \sum_{k=1}^{n} M_k(f+g)\Delta\alpha_k \leqslant \sum_{k=1}^{n} M_k(f)\Delta\alpha_k + \sum_{k=1}^{n} M_k(g)\Delta\alpha_k$$

(i.e.) $U(P,(f+g),\alpha) \leq U(P,f,\alpha) + U(P,g,\alpha)$

Taking infimum, we get, $\int_a^{-b} (f+g)d\alpha \leq \int_a^{-b} fd\alpha + \int_a^{-b} gd\alpha.$

Similarly, we can prove that $\int_{-a}^{b} (f+g)d\alpha \geqslant \int_{-a}^{b} fd\alpha + \int_{-a}^{b} gd\alpha$

Riemann's Condition:

Definition 2.28:

We say that 'f' satisfies Riemann's condition with respect to $'\alpha'$ on [a,b] if for every $\varepsilon > 0$, $\exists$ a partition P_ε such that P finer than P_ε implies $0 \leq U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Theorem 2.29:

Assume that $\alpha \nearrow [a,b]$. Then the following three statements are equivalent

(i) f $\in R(\alpha)$ on [a, b].

(ii) f satisfies Riemann's condition w.r.to α on [a. b]

(iii) I (f, $\alpha) = I(f,\alpha)$

Proof:

Let f and $'\alpha'$ be real – valued fns defined on [a, b].

Given α is an increasing on [a, b]

(i) TP: $(i) \to (ii)$

Assume that f $\in R(\alpha)$ on [a,b]

Let P= {a=$x_0, x_1, \dots, x_n = b\} \in P[a,b]$

Let $t_k \in [x_{k-1}, x_k]$ and $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given

To prove: f satisfies Riemann's condition w.r.to α on [a,b]

(i.e.) To prove: 0≤ $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Here 'α' is increasing on [a, b]

Case (i): $\alpha(a) = \alpha(b)$

'α' is a constant function.

$\therefore \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = 0$

(i.e.) $\Delta\alpha_k = 0 \ \forall\ k = 1,2, \dots, n$

$U(P, f, \alpha) = 0$ and $L(P, f, \alpha) = 0$

$\Rightarrow 0 = U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

∴ Riemann's condition is satisfied trivially

Case (ii): $\alpha(a) < \alpha(b)$

Now, f ∈ $R(\alpha)$

$\Rightarrow \exists\ A \in \mathbb{R}\ \ni: \forall\ \varepsilon_1 = \frac{\varepsilon}{3} > 0, \exists\ P_{\varepsilon_1}$ of [a, b] ∋ ∀ P is finer than P_{ε_1} and all choice of $t_k, t_k' \in [x_{k-1}, x_k]$ we have,

$$\left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A\right| < \frac{\varepsilon}{3}$$

$$\left|\sum_{k=1}^{n} f(t_k')\Delta\alpha_k - A\right| < \frac{\varepsilon}{3} \qquad \dots\dots\dots\dots \text{(i)}$$

Where A = $\int_a^b f\, d\alpha$

Now,

$$\left|\sum_{k=1}^{n} (f(t_k) - f(t_k'))\Delta\alpha_k\right| = \left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \sum_{k=1}^{n} f(t_k')\Delta\alpha_k\right|$$

$$= \left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A + A - \sum_{k=1}^{n} f(t_k')\Delta\alpha_k\right|$$

$\leq |\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - A| + |-(\sum_{k=1}^{n} f(t_k')\Delta\alpha_k - A)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ [by 1]

$= \frac{2\varepsilon}{3}$

$|\sum_{k=1}^{n}(f(t_k) - f(t_k'))\Delta\alpha_k| < \frac{2\varepsilon}{3}$

Now, $M_k(f) - m_k(f) = \sup f(x) - \inf f(x) \forall\, x \in [x_{k-1}, x_k]$

$= \sup f(x) - \sup f(-x)$

$= \sup f(x) - \sup f(x')\ \forall\ x, x' \in [x_{k-1}, x_k]$

$\therefore M_k(f) - m_k(f) = \sup\{f(x) - f(x') : x, x' \in [x_{k-1}, x_k]\}$

$\geq f(x) - f(x')$

$\therefore M_k(f) - m_k(f) \geq f(x) - f(x')\ \forall\ h > 0, \exists\ t_k, t'_k \in [x_{k-1}, x_k] \ni:$

$M_k(f) - m_k(f) - h < f(t_k) - f(t'_k)$

$\Rightarrow M_k(f) - m_k(f) < f(t_k) - f(t'_k) + h$

Choose h = $\frac{\varepsilon}{3[\alpha(b)-\alpha(a)]} > 0$

Now, $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{k=1}^{n}(f) M_k\, \Delta\alpha_k - \sum_{k=1}^{n}(f) m_k\, \Delta\alpha_k$

$= \sum_{k=1}^{n}[(f)M_k - m_k(f)]\Delta\alpha_k$

$< \sum_{k=1}^{n}[(f)M_k - m_k(f)]\Delta\alpha_k$

$< \sum_{k=1}^{n}[f(t_k) - f(t_k') + h]\ \Delta\alpha_k$

$= \sum_{k=1}^{n}[f(t_k) - f(t_k')]\ \Delta\alpha_k + \sum_{k=1}^{n} h\Delta\alpha_k$

$< \frac{2\varepsilon}{3} + \sum_{k=1}^{n} h\Delta\alpha_k$ (by 2)

$= \frac{2\varepsilon}{3} + \frac{\varepsilon}{3(\alpha(b)-\alpha(a))} \alpha(b) - \alpha(a)$

$= \varepsilon$

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

∴ Riemann's condition is satisfied.

(ii) To prove: (ii)⟹(iii)

Assume that 'f' satisfies Riemann's condition w.r.to s on [a, b].

$\Longrightarrow \forall\, \varepsilon > 0, \exists$ a partition P_ε ∋: P finer than P_ε

$\Longrightarrow 0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

To prove: $\underline{I}\,(f, \alpha) = \bar{I}\,(f, \alpha)$

Now, $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

$\Longrightarrow\ U(P, f, \alpha) < L(P, f, \alpha) + \varepsilon$ ………… (3)

We know that, $inf U(P, f, \alpha) < U(P, f, \alpha)$

$\Longrightarrow \bar{I}$ (f, α) $<\ U(P, f, \alpha)$ ………… (4)

And we know that,

$L(P, f, \alpha) < sup L(P, f, \alpha)$

$\Longrightarrow L(P, f, \alpha) < \underline{I}(f, \alpha)$ ……… (5)

Now,

$\bar{I}$ (f, α) $<\ U(P, f, \alpha)$ [by equation (4)]

$< L(P, f, \alpha) + \varepsilon$ [by equation (3)]

$<$ $\underline{I}$ (f, α)+ ε [by equation (5)]

$\bar{I}$ (f, α) $<$ $\underline{I}$ (f, α) $+ \varepsilon\ \forall\ \varepsilon > 0$

Since, $\varepsilon > 0$ is arbitrary,

$\underline{I}\,(f, \alpha) \leq\ \bar{I}\,(f, \alpha)$ ………(6)

Given $\alpha \nearrow$ on [a, b]

Then by Theorem 2.24,

I (f, α) ≤ I (f, α) ……….(7)

From equation (6) and (7) we get,

I (f, α) $\leq$ I (f, α)

(iii) $\Longrightarrow (i)$

Assume that I (f, α) $=$ I (f, α)

To prove: f $\in R(\alpha)$ on [a, b]

(i.e.) $\int_a^b f\,d\alpha$ exists

(i.e.) To prove: $|S(P, f, \alpha) - A| < \varepsilon$ where A = $\int_a^b f\,d\alpha$

We know that $\bar{I}\,(f, \alpha) =$ inf{ $U(P, f, \alpha): P \in \wp[a, b]$}

$\underline{I}\,(f, \alpha) =$ sup{ $L(P, f, \alpha): P \in \wp[a, b]$}

Given $\bar{I}\,(f, \alpha) = \underline{I}\,(f, \alpha)$………………. (8)

Given, $\varepsilon > 0$ choose $P_\varepsilon' \ni$: $U(P, f, \alpha) < \bar{I}\,(f, \alpha) + \varepsilon\ \forall\ P\ finer\ than\ P_\varepsilon'' \ni$:

$L(P, f, \alpha) < \underline{I}\,(f, \alpha) + \varepsilon\ \forall\ P\ finer\ than\ P_\varepsilon''$

Let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$

Then $\forall$ P finer than P_ε

$\underline{I}\,(f, \alpha) - \varepsilon < L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha) < \bar{I}\,(f, \alpha) + \varepsilon$

$\Longrightarrow \underline{I}\,(f, \alpha) - \varepsilon < S(P, f, \alpha) \leq U(P, f, \alpha) < \bar{I}\,(f, \alpha) + \varepsilon$

$\Longrightarrow A - \varepsilon < S(P, f, \alpha) < A + \varepsilon$

$\Longrightarrow -\varepsilon < S(P, f, \alpha) - A < \varepsilon$

$\Longrightarrow |S(P, f, \alpha) - A| < \varepsilon$

Hence f $\in R(\alpha)$ on [a, b]

Comparison Theorems:

Theorem 2.30:

Assume that $\alpha \nearrow$ on [a,b]. If f $\epsilon\, R(\alpha)$ and g $\epsilon\, R(\alpha)$ on [a,b] and if f(x)≤ g(x) for all x in [a,b], then we have $\int_a^b f(x)d\alpha(x) \leq \int_a^b g(x)d\alpha(x)$.

Proof:

Given $\alpha \nearrow$ on [a,b].

Let P = {a = $x_0, x_1, \ldots, x_n = b\} \,\epsilon$ D[a,b]

Let $t_k \,\epsilon\, [\, x_{k-1},\, x_k]$ and $\Delta\, \alpha_k = \alpha(\, x_k) - \alpha(\, x_{k-1})$

Let $\epsilon > 0$ be given

f $\epsilon\, R(\alpha)$ and g $\epsilon\, R(\alpha)$ on [a,b] and if f(x)≤ g(x) for all x in [a,b],

To prove: $\int_a^b f(x)d\alpha(x) \leq \int_a^b g(x)d\alpha(x)$

$\alpha \nearrow$ on [a, b],

$\Delta\, \alpha_k$ for all k = 1,2, …. n

Given f(x) ≤ g(x) for all x ∈ [a,b]

f(t_k) ≤ g(t_k) for all $t_k \in [x_{k-1}, x_k]$

$$\sum_{k=1}^{n} f(t_k)\, \Delta\, \alpha_k \leq \sum_{k=1}^{n} g(t_k)\, \Delta\, \alpha_k$$

S(p, f, α) ≤ S(p,g, α) … … … (3)

f,g ∈ $R(\alpha)$ for $\|P\| \to 0$, we have S(p,f, α) $\to \int_a^b f d\alpha$ and S(p,g, α) $\to \int_a^b g d\alpha$

from equation (3) $\int_a^b f d\alpha \leq \int_a^b g d\alpha$

(i.e.), $\int_a^b f(x)d\,\alpha(x) \leq \int_a^b g(x)d\,\alpha(x)$

Note:

In particular, the above Theorem implies that whenever $g(x) \geq 0$ and $\alpha \nearrow$ on [a,b], $\int_a^b g(x)d\,\alpha(x) \;\geq 0$

Theorem 2.31:

Assume that $\alpha \nearrow$ on [a,b]. |f f ϵ $R(\alpha)$ on [a,b], then |f| ϵ $R(\alpha)$ on [a,b] and we have the inequality $|\int_a^b f(x)d\,\alpha(x)| \leq \int_a^b |f(x)|d\,\alpha(x)$.

Proof:

Assume that $\alpha \nearrow$ on [a,b].

Let P={a= $x_0, x_1, \ldots, x_n = b\}$ ϵ D[a,b]

Let $t_k \;\epsilon\; [\, x_{k-1},\; x_k]$ and $\Delta\,\alpha_k = \alpha(\,x_k) - \alpha(\,x_{k-1})$

Let $\epsilon > 0$ be given

Given f ϵ $R(\alpha)$ on [a,b]

Given $\epsilon > 0$, there exist a partition P_ε such that P finer than P_ε

U(P, f, α) $- L$(P,f, α) $< \;\varepsilon$

$$\sum_{k=1}^{n} M_k(\mathrm{f})\,\Delta\,\alpha_k - \sum_{k=1}^{n} m_k(\mathrm{f})\,\Delta\,\alpha_k < \;\varepsilon$$

$\sum_{k=1}^{n}[M_k(\mathrm{f}) - m_k(\mathrm{f})]\,\Delta\,\alpha_k < \;\varepsilon$-------(1)

Where, $M_k(\mathrm{f}) =$ sup {f(x): x∈[$x_{k-1},\; x_k$]} and

$m_k(\mathrm{f}) =$ inf {f(x): x∈[$x_{k-1},\; x_k$]}

To prove: |f| ϵ $R(\alpha)$

Now, $M_k(\mathrm{f}) - m_k(\mathrm{f})$= sup f(x) – inf f(x), x ∈[$x_{k-1},\; x_k$]

= sup f(x) – sup f(-x)

= sup f(x) – sup f(y)

$M_k(\mathrm{f}) - m_k(\mathrm{f})$=sup {f(x) – f(y) : x,y ∈[$x_{k-1},\; x_k$]}----(2)

We know that ||f(x) – f(y)|| ≤||f(x) – f(y)||

Equation (2)=> $|M_k(|f|) - m_k(|f|)$=|sup {|f(x)| – |f(y)| : x,y ∈[x_{k-1}, x_k]}

≤|sup {(f(x) – f(y)) : x,y ∈[x_{k-1}, x_k]}|

= $|M_k(f) - m_k(f)|$

$|M_k(|f|) - m_k(|f|) \leq |M_k(f) - m_k(f)|$

Sup{f(x) – f(y) : x,y ∈[x_{k-1}, x_k]} > 0

$|M_k(|f|) - m_k(|f|) \leq |M_k(f) - m_k(f)|$

$$\sum_{k=1}^{n} [M_k(|f|) - m_k(|f|)] \Delta \alpha_k \leq \sum_{k=1}^{n} [|M_k(f) - m_k(f)|] \Delta \alpha_k$$

$\sum_{k=1}^{n} M_k(|f|)\Delta \alpha_k - m_k(|f|) \Delta \alpha_k < \varepsilon$ (by (1))

$U(P, |f|, \alpha) - L(P,|f|, \alpha) < \varepsilon$

f $\epsilon R(\alpha)$ on [a,b]

Take g = |f|

Then by Theorem 2.30, $\int_a^b f(x) d\alpha(x)| \leq \int_a^b |f(x)| d\alpha(x)$.

$-\int_a^b f(x) d\alpha(x) \leq \int_a^b f(x) d\alpha(x) \leq \int_a^b |f(x)| d\alpha(x)$.

$\int_a^b f(x) d\alpha(x) \leq \int_a^b |f(x)| d\alpha(x)$

Note:

The converse of the above Theorem is not true. (i.e.) $\alpha \nearrow$ on [a,b] and |f| ∈ $R(\alpha) \not\Rightarrow$ f ∈ $R(\alpha)$

Theorem 2.32:

Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, then $f^2 \in R(\alpha)$ on $[a, b]$.

Proof:

Assume that $\alpha \nearrow$ on $[a, b]$

Let $P = \{a = x_0, x_1, \dots, x_n = b\} \in \mathcal{P}_2[a, b]$

Let $t_k \in [x_{k-1}, x_k] + \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given

Let M be an upper bound of |f| on $[a, b]$ …… (1)

Given $f \in R(\alpha)$ on $[a, b] \Rightarrow |f| \in R(\alpha)$ on $[a, b]$ (by Thm (7.2))

$\Rightarrow \forall \varepsilon_1 = \frac{\epsilon}{QM} > 0$, $\exists$ a partition $P_{\varepsilon_1} \in P[a, b]$:

P is finer than P_{ε_1} implies.

$U(P_2, |f|, \alpha) - L(P_{1,}|f|, \alpha) < \varepsilon_1$

$$\Rightarrow \sum_{k=1}^{n} M_k(|f|)\Delta\alpha_k - \sum_{k=1}^{n} m_k(|f|)\Delta\alpha_k < \frac{\varepsilon}{2M}$$

$$\Rightarrow \sum_{k=1}^{n} [M_k(|f\ |) - m_k(|f\ |)]\Delta\alpha_k < \frac{\varepsilon}{\Delta M} \qquad \dots..(2)$$

Where, $M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$and\ m_k(f) = \ln f\{f(x) : x \in [x_{k=1,2}x_k]\}$

To prove: $f^2 \in R(\alpha)$ on $[a, b]$

Now,

$$\begin{aligned} M_k(f^2) &= \sup\{[f(x)]^2 : x \in [x_{k-1}, x_k]\} \\ &= \sup\{|f(x)|^2 : x \in [x_{k-1}, x_k]\} \\ &= [\sup\{|f(x)| : x \in [x_k, x_k]\}]^2 \\ &= [M_k(|f|)]^2 \end{aligned}$$

(i.e.), $M_k(f^2) = [M_k(|f|)]^2$
similarly $m_k(f^2) = [m_k(|f|)]^2$

Now,

$M_k(f^2) - m_k(f^2) = [M_k(|f|]^2 - [m_k(|f|)]^2$
$= (M_k(|f|) + m_k(|f|))(M_k|f|) - m_k(|f|))$
$m_k(f^2) - m_k(f^2) \leq 2M[M_k(|f|) - m_k(|f|)]$ (by(1))

$\Rightarrow \sum_{k=1}^{n} M_k(f^2).\Delta\alpha_k - \sum_{k=1}^{n} m_k(f^2).\Delta\alpha_k$

$\leqslant 2M[\sum_{k=1}^{n}[M_k(|f|) - m_k(|f|)]\Delta\alpha_k]$

$\Rightarrow U(P, f^2, \alpha) - L(P, f^2, \alpha) < 2M \cdot \frac{\varepsilon}{2M} -$ (by (2))

$\Rightarrow U(P, f^2, \alpha) - L(P, f^2, \alpha) < \varepsilon \Rightarrow f^2 \in R(\alpha)$ on $[a, b]$.

Theorem 2.33:

Assume that $\alpha \nearrow$ on $[a, b]$. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$, then the product $f.g \in R(\alpha)$ on $[a, b]$.

Proof:

Assume that $\alpha \nearrow$ on $[a, b]$

Given $f \in R(\alpha) \& g \in R(\alpha)$ on $[a, b]$

To prove: $f.g \in R(\alpha)$ on $[a, b]$

Now, $[f(x) + g(x)]^2 = [f(x)]^2 + [g(x)]^2 + 2f(x)g(x)$

$\Rightarrow f(x)g(x) = \frac{1}{2}[f(x) + g(x)]^2 - \frac{1}{2}[f(x)]^2 - \frac{1}{2}[g(x)]^2$

$\because f \in R(\alpha)$ *and* $g \in R(\alpha)$, by Theorem2.4 & 2.32,

$\frac{1}{2}[f + g]^2, \frac{1}{2} \cdot f^2, \frac{1}{2} \cdot g^2 \in R(\alpha)$

=> $\frac{1}{2}(f + g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2 \in R(\alpha)$ (i.e.) $f.g \in R(\alpha)$

Unit III

The Riemann-Stieltjes Integral - Integrators of bounded variation-Sufficient conditions for the existence of Riemann-Stieltjes Integrals-Necessary conditions for the existence of RS integrals- Mean value theorems -integrals as a function of the interval –Second fundamental Theorem of integral calculus-Change of variable -Second Mean Value Theorem for Riemann integral- Riemann-Stieltjes integrals depending on a parameter.

INTEGRATORS OF BOUNDED VARIATIONS

Note 3.1:

If 'α' is of bounded variation on [a, b], then 'α' can be expressed as the difference of two increasing functions α_1 and α_2. (i.e.). $\alpha = \alpha_1 - \alpha_2$.

If $\alpha = \alpha_1 - \alpha_2$ is such a decomposition and if f ∈ R(α_1) and f ∈ R(α_2) on [a, b] , then f ∈ R(α)

But the converse is not true.

(i.e.). If f ∈ R(α) on [a, b], then it is quite possible to choose increasing functions α_1 and α_2 such that neither integral $\int_a^b f\, d\alpha_1$ and $\int_a^b f\, d\alpha_2$ exits

The uniqueness of the decomposition $\alpha = \alpha_1 - \alpha_2$ is not possible .

The converse is true when there exists at least one decomposition such that α_1 is the total variations of α *and* $\alpha_2 = \alpha_1 - \alpha$.

Theorem 3.2:

Assume that α is the bounded variations on [a, b]. Let V(x) denote the total variation of 'α' on [a, x]. If a< x ≤ b , and let V(a) = 0. Let f be defined and bounded on [a, b]. If f ∈ R(α) on [a, b], then f ∈ R(V) on [a, b].

Proof:

Let α be of bounded variation on [a, b]

Let V:[a, b]→ $\mathcal{R}$ such that

$$V(x) = \begin{cases} 0 & if\ x = a \\ V_\alpha(a,x) & if\ a < x \le b \end{cases} \quad \ldots\ldots\ldots (1)$$

Let P = {a= x_0, x_1, ... x_n= b}∈ $\mathcal{P}$[a, b]

Let t_k ∈ [x_{k-1}, x_k] and $\Delta\alpha_k$= α(x_k) – α(x_{k-1})

Let ε >0 be given

Let f be defined and bounded on [a, b]

⇒There exits M > 0 such that | $f(x)$ |< M, where x ∈ [0, b] ……..(2)

Given: f ∈ R (α) on [a, b]

To prove: f ∈ R(V) on [a, b]

Case (i): V(b) = 0

⇒V is a constant function => $\Delta V_k = 0$

⇒f ∈ R(V)

case(ii) V(b)> 0

⇒from the definition of total variation , we get x < y ⇒ V(x) ≤ V(y)

⇒V is an increasing function on [a, b]

Therefore, it is enough to show that, 'f' satisfies the Riemann condition w.r.to V on [a, b], (i.e.)To prove: U(p, f , v) – L(p, f, v) < ε

Now, Given: f ∈ R (α)

⇒There exists A ∈ $\mathcal{R}$ suchthat ∀ $\varepsilon_1 = \frac{\varepsilon}{8} > 0$, there exist P_{ε_1} of [a, b] suchthat

∀ P finer than & all choice of t_k , t_k' ∈ [x_{k-1} , x_k],

We have

$|\sum_{k=1}^{n} f(t_k)\,\Delta\,\alpha_k$- A$| < \frac{\varepsilon}{8}$ & $|\sum_{k=1}^{n} f(t_k')\,\Delta\,\alpha_k$- A$| < \frac{\varepsilon}{8}$ where A= $\int_a^b f\, d\alpha$

Now,

$|\sum_{k=1}^{n}(f(t_k)\, f(t_k'))\,\Delta\,\alpha_k\,| = |\sum_{k=1}^{n} f(t_k)\,\Delta\,\alpha_k + \sum_{k=1}^{n} f(t'_k)\,\Delta\,\alpha_k\,|$

$=|\sum_{k=1}^{n} f(t_k)\,\Delta\,\alpha_k - A + A - \sum_{k=1}^{n} f(t'_k)\,\Delta\,\alpha_k\,|$

$\leq|\,\sum_{k=1}^{n} f(t_k)\,\Delta\,\alpha_k - A| + |-(\sum_{k=1}^{n} f(t_k)\,\Delta\,\alpha_k - A)|$

$< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$

Therefore, $|\sum_{k=1}^{n}(f(t_k)\,f(t_k'))\,\Delta\,\alpha_k\,| < \frac{\varepsilon}{4}$ ……... (3)

Now,

V(b) = V_α(a, b) (by 1)

$= \sup\{\sum_{k=1}^{n} |\,\Delta\alpha_k\,|\}$

$\geq \sum_{k=1}^{n} |\,\Delta\alpha_k\,|$

(i.e.) V(b)> $\sum_{k=1}^{n} |\,\Delta\alpha_k\,|$ ………(4)

By the property of supremum,

$\Rightarrow V(b) - \frac{\varepsilon}{4m} < \sum_{k=1}^{n} |\,\Delta\alpha_k\,|$

$V(b) - \sum_{k=1}^{n} |\,\Delta\alpha_k\,| < \frac{\varepsilon}{4m}$ ………(5)

Now, we note that $\Delta\,V_k - |\Delta\,\alpha_k\,| \geq 0$

Therefore,

$\sum_{k=1}^{n}[M_k(f) - m_k(f)](\Delta\,V_k - |\Delta\,\alpha_k\,|) \leq \sum_{k=1}^{n}(m - (-m)(\Delta\,V_k - |\Delta\,\alpha_k\,|)$

$= 2m\,(\sum_{k=1}^{n}\Delta\,V_k - \sum_{k=1}^{n} |\,\Delta\,\alpha_k\,|)$

$=2m(V(b) - V(0) - \sum_{k=1}^{n} |\,\Delta\,\alpha_k\,|)$

$=2m(V(b) - \sum_{k=1}^{n} |\,\Delta\,\alpha_k\,|)$

$<2m\,\frac{\varepsilon}{4m} = \frac{\varepsilon}{2}$ (by equation (5))

Therefore, $\sum_{k=1}^{n}[M_k(f) - m_k(f)](\Delta\,V_k - |\Delta\,\alpha_k\,|) < \frac{\varepsilon}{2}$ …………(6)

Let $A(p)= \{k: \Delta\alpha_k \geq 0\}$

& $B(p)=\{k: \Delta\alpha_k < 0\}$

Let h= $\frac{\varepsilon}{4v(b)}$ >0 ………… (7)

We know that, $M_k(f)- m_k(f)= \sup f(x) – \inf f(x), x \in [x_{k-1}, x_k]$

$=\sup f(x) –\sup (-f(x))$

$= \sup f(x) – \sup (f(y)), x, y \in[x_{k-1}, x_k]$

$M_k(f)- m_k(f)=\sup \{ f(x)- f(y) ; x, y \in[x_{k-1}, x_k]$

$M_k(f)- m_k(f) \geq f(x) – f(y)$

If $k \in A(p)$, choose t_k & t_k' such that $M_k(f)- m_k(f) –h < f(t_k)- f(t_k')$

$\Rightarrow M_k(f)- m_k(f) < f(t_k)- f(t_k')+ h$

If, If $k \in B(p)$, choose t_k & t_k' such that $M_k(f)- m_k(f) –h < f(t'_k)- f(t_k)$

$\Rightarrow M_k(f)- m_k(f) < f(t_k') – f(t_k) + h$

Now, $\sum_{k=1}^{n}\left(M_k(f) - m_k(f)\right) \mid \Delta\alpha_k \mid < \sum_{k\in A(p)}(f(t_k) - f(t_k') + h) \mid \Delta\alpha_k \mid$

$+ \sum_{k\in B(p)} f\left((t_k')\right) - f(t_k) + h) \mid \Delta\alpha_k \mid$

$=\sum_{k\in A(p)}(f(t_k) - f(t_k') + h) \mid \Delta\alpha_k \mid + \sum_{k\in B(p)} f\left((t_k')\right) - f(t_k) + h) \mid \Delta\alpha_k \mid$

$+ \sum_{k\in A(p)} h \mid \Delta\alpha_k \mid + \sum_{k\in A(p)} h \mid \Delta\alpha_k \mid$

$=\sum_{k\in A(p)}(f(t_k) - f(t_k')) \Delta\alpha_k + \sum_{k\in B(p)} f\left((t_k')\right) - f(t_k)) \Delta\alpha_k + \sum_{k=1}^{n} h. \mid \Delta\alpha_k \mid$

$=\sum_{k=1}^{n}(f(t_k) - f(t_k')) \Delta\alpha_k + h \sum_{k=1}^{n} \mid \Delta\alpha_k \mid$

$< \frac{\varepsilon}{4} + h[v(b)]$ (by equation (5) & (4))

$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4 .v(b)}. V(b)$ (by equation (7))

$= \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$

$\sum_{k=1}^{n}[M_k(f) - m_k(f)] \mid \Delta\alpha_k \mid < \frac{\varepsilon}{2}$ ………. (8)

Now,

$\sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta v_k = \sum_{k=1}^{n}[M_k(f) - m_k(f)](\Delta v_k - \mid \Delta\alpha_k \mid + \mid \Delta\alpha_k \mid)$

$= \sum_{k=1}^{n}[M_k(f) - m_k(f)](\Delta v_k - |\Delta\alpha_k|) + \sum_{k=1}^{n}[M_k(f) - m_k(f)]\,|\Delta\alpha_k|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ (by equation (6) & (7))

Therefore, $\sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta v_k < \varepsilon$

$\Rightarrow \sum_{k=1}^{n} M_k(f)\Delta v_k - \sum_{k=1}^{n} m_k(f)\Delta v_k < \varepsilon$

$\Rightarrow$U(P, f, v) – L(P, f, v) $< \varepsilon$

$\Rightarrow$f ∈ R(V) on [a, b]

Theorem 3.3:

Let α be of bounded variations on [a, b] and assume that f ∈ R(α) on [a, b]. Then f ∈ R(α) on every subinterval [c, d] of [a, b]

Proof:

Let α be a bounded variation on [a, b]

Let v:[a, b]→$\mathcal{R}$ such that

$$V(x) = \begin{cases} 0 & if\ x = a \\ v_\alpha(a,x) & if\ a < x \le b \end{cases} \quad \ldots\ldots\ldots\ldots(1)$$

Let P= { a= x_0 ,x_1 ....x_n =b} ∈ $\mathcal{P}$(a, b)

Let $\varepsilon > 0$ be given

Given f ∈ R(α) on [a, b]

To prove: f ∈ R(α) on [c, d] of [a, b]

∴, α is of bounded variation on [a, b], α = V – (V- α)where V & V- α are ↗ on [a, b]

Then by Theorem 3.2,

f ∈ R(α) on [a, b] => f ∈ R(v) & f ∈ R(V- α) on [a, b]

$\Rightarrow$f ∈ R(v) & f ∈ R(V- α) on [c, d]   since (α & V- α are ↗ on [a, b])

$\Rightarrow$f ∈ R(α) on [c, d]   since (α = v-(v-α)

$\therefore$ f $\in$ R(α) on [a, b] $\Rightarrow$ f $\in$ R(α) on [c, d]

Now,

We shall prove that Theorem when α is increasing on [a, b].

Assume that a < c < b

By Theorem 2.7. of integration by parts,

$\int_a^d f\, d\alpha = \int_a^c f\, d\alpha + \int_c^d f\, d\alpha$ where $a \leq c \leq d \leq b$

$\Rightarrow \int_c^a f\, d\alpha = \int_a^d f\, d\alpha - \int_a^c f\, d\alpha$

To prove: f $\in$ R(α) on [c, d]

(i.e.), To prove: 'f' satisfies the Riemann condition w.r.to α on [c, d]

(i.e.), To prove: $\int_c^d f\, d\alpha$ exists.

(i.e.), To prove: $\int_a^d f\, d\alpha$ & $\int_a^c f\, d\alpha$ exist

Since, f $\in$ R(α) on [a, b]of [a, b]

Given: $\varepsilon > 0$, there exist a position P_ε on [a, b] such that P is finer than $P_\varepsilon$implies U(P, f, α)- L(P, f, α) $< \varepsilon$

Let Δ(p, x) = U(P, f, α)- L(P, f, α) on [a, x]

$\therefore$ *equation* (2) $\Rightarrow$ Δ(p, b) $< \varepsilon$……….(3)

Assume that c $\in P_\varepsilon$

Let P_ε' be a partition of P_ε on [a, c]

Let P' be a partition finer than P_ε' on [a, c]

(i.e.)., $P' \supseteq P_\varepsilon'$

Then P= $P' \cup P_\varepsilon'$ Is a partition of [a, b]

(i.e.) P composed of the points of P' along with those points of P_ε in [a, b]

From equation (3) $\Longrightarrow$

$\varepsilon > \Delta(p, b)$

$=\Delta(P', c) + \Delta(P_\varepsilon, b)$

$> \Delta(\mathrm{P'}, \mathrm{C})$

Therefore, $\Delta(P', c) < \varepsilon$

(i.e.)., U(P', f, α)- L(P', f, α)$<$ ε on [a, c]

(i.e.)., f satisfies a Riemann condition on [a, c]& $\int_a^c f\, d\alpha$ exists

Similarly, $\int_c^d f\, d\alpha$ exists

(i.e.) f $\in$ R(α) on [c, d]

Theorem 3.4:

Assume f $\in$ R(α) & g $\in$ R(α) on [a, b] , where $\alpha \nearrow$ on [a, b] . Define

F(x)= $\int_a^x f(t) d\alpha(t)$ and G(x)= $\int_a^x g(t) d\alpha(t)$ if x$\in$ [a, b], then f $\in$ R(G) ,

g $\in$ R(F), and the product f g $\in$ R(α) on [a, b], and we have

$$\int_a^b f(x) g(x) d\alpha(x) = \int_a^b f(x) dG(x) = \int_a^b g(x) dF(x)$$

Proof:

Let $\alpha \nearrow$ on [a, b]

Let P= {a=x_0 , x_1,……….x_n=b} $\in \mathcal{P}$[a, b]

Let $t_k \in [x_{k-1}, x_k]$ & $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given

Assume that f $\in R(\alpha)$ & g $\in R(\alpha)$ on [a, b]

Define F(x) = $\int_a^x f(t) d\alpha(t)$ & G(x) = $\int_a^x g(t) d\alpha(t)$ if x $\in [a, b]$

To prove: f $\in$ R(G) , g $\in$ R(F), f g $\in$ R(α)

$$\int_a^b f(x) g(x) d\alpha(x) = \int_a^b f(x) dG(x) = \int_a^b g(x) dF(x)$$

Here given f & g ∈ R(α) on [a, b]

Then by theorem 2.33, f. g ∈ R(α) on [a, b]

Now, To prove: f ∈ R(α) & $\int_a^b f(x)g(x)d\alpha(x)$= $\int_a^b f(x)dG(x)$

Let M_g = sup {|g(x)| : x ∈ [a, b]}

Given f ∈ $R(\alpha)$

⟹For all $\varepsilon > 0$, there exit a partition P_ε such that P finer than P_ε implies U(P, f, α)- L(P, f, α) $< \frac{\varepsilon}{Mg}$......1

For any partition P

S(P, f, G)= $\sum_{k=1}^{n} f(t_k)\Delta G_k$

$=\sum_{k=1}^{n} f(t_k)[G(x_k) - G(x_{k-1})]$

$= \sum_{k=1}^{n} f(t_k)[\int_a^{x_k} g(t)d\alpha(t)$ - $\int_a^{x_{k-1}} g(t)d\alpha(t)]$

$=\sum_{k=1}^{n} f(t_k)[\int_a^{x_k} g(t)d\alpha(t)$ + $\int_{x_{k-1}}^{a} g(t)d\alpha(t)]$

$= \sum_{k=1}^{n} f(t_k)[\int_{x_{k-1}}^{x_k} g(t)d\alpha(t)$

S(P, f, G)= $\sum_{k=1}^{n} f(t_k)[\int_{x_{k-1}}^{x_k} g(t)d\alpha(t)$............(2)

We can write, $\int_a^b f(x)g(x)d\alpha(x)$= $\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)g(t)d\alpha(t)$...........(3)

Equation (2) – (3) ⟹ S(P,f,G)-$\int_a^b f(x)g(x)d\alpha(x)$

$=\sum_{k=1}^{n} f(t_k)[\int_{x_{k-1}}^{x_k} g(t)d\alpha(t)$ - $\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)g(t)d\alpha(t)$

$=\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [f(t_k)$- f(t)]g(t) d$\alpha$(t)

∴ | S(P, f, G) - $\int_a^b f(x)g(x)d\alpha(x)$| = |$\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [f(t_k)$- f(t)]g(t) d$\alpha$(t)|

$\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k}$ | $f(t_k)$- f(t)||g(t)| | dα(t)|

$=M_g \int_{x_{k-1}}^{x_k} [M_k(f) - m_k(f)]d\alpha(t)$

$=M_g\{\int_a^b M_k(f)d\alpha(t) - \int_a^b m_k\,(f)d\alpha(t)\}$

$=M_g\,\{U(P, f, \alpha)- L(P, f, \alpha)\}$

$<M_g\,\frac{\varepsilon}{M_g}$ (by 1)

$=\varepsilon$

$\therefore |\,S(P, f, G) - \int_a^b f(x)g(x)d\alpha(x)| < \varepsilon$

$\therefore$ $f \in R(G)$ & $\int_a^b f(x)dG(x) = \int_a^b f(x)g(x)d\alpha(x)$

Similarly, we can prove that $f \in R(F)$ & $\int_a^b f(x)dF(x) = \int_a^b f(x)g(x)d\alpha(x)$

Sufficient conditions for Existence of Riemann – Stieltjes integrals

Theorem 3.5:

If 'f' is continuous on [a, b] and if 'α' is of bounded variation on[a,b] then $f\in R(\alpha)$ on $[a, b]$

Proof:

Given f is continuous on [a, b]

f is bounded on [a, b]

Given 'α' is of bounded variation on[a,b] then

Let V: [a, b]→ ℝ such that

$$V(x) = \begin{cases} 0 & if\ x = a \\ v_\alpha(a, x) & if\ a < x \le b \end{cases}$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\} \in \mathcal{P}[a, b]$

Let $t_k \in [x_{k-1}, x_k]$ & $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Let $\varepsilon > 0$ be given

To prove: $f\in R(\alpha)$ on $[a, b]$

[$\therefore$ α is of bounded variation on[a,b] , then $\alpha = v - (v - \alpha)$ where v& v- α are ↗on [a,b]

$\therefore$ $f\in R(v)$ & $f\in R(v-\alpha)$ *on [a,b]*

$\Longrightarrow$ f∈R(α) on $[a, b]$]

now, we shall prove that theorem when $\alpha \nearrow$on [a,b]

(i.e.) a<b $\Longrightarrow \alpha(a) \leq \alpha(b)$

suppose, $\alpha(a) = \alpha(b)$ then,

$\Delta\alpha_k = 0$

∴U(p,f, α) -L(p,f, α) =0< ε

∴f∈R(α) on $[a, b]$

suppose, $\alpha(a) < \alpha(b)$

Given f is continuous on [a, b]

$\Longrightarrow$ f is uniformly continuous on [a, b]

$\Longrightarrow$ given $\varepsilon > 0$ there exists δ >0 such that

$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon/A$ ……..(1)

Where, A= 2[α (b) – α (a)]

Let P_ε be a partition of [a,b] such that $\|P_\varepsilon\| < \delta$

If P is finer than P_ε, *then* $\|P\| < \delta$

∴ $|f(t_k) - f(t'_k)| < \frac{\varepsilon}{2(\alpha(b)-\alpha(a))}$, $t_k, t'_k \in [x_{k-1}, x_k]$

Now, $M_k(f) - m_k(f) = \sup\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$

$= |f(t_k) - f(t'_k)|$ there exists , $t_k, t'_k \in [x_{k-1}, x_k]$

$< \frac{\varepsilon}{2(\alpha(b)-\alpha(a))}$

∴$M_k(f) - m_k(f) < \frac{\varepsilon}{2(\alpha(b)-\alpha(a))}$

$$\Longrightarrow \sum_{k=1}^{n}[M_k(f) - m_k(f)]\Delta\alpha_k < \frac{\varepsilon}{2(\alpha(b)-\alpha(a))}\sum_{k=1}^{n}\Delta\alpha_k$$

$\Longrightarrow \sum_{k=1}^{n} M_k(f)\Delta\alpha_k$ - $\sum_{k=1}^{n} m_k(f)\Delta\alpha_k < \frac{\varepsilon}{2(\alpha(b)-\alpha(a))}(\alpha(b) - \alpha(a))$

$\therefore \sum_{k=1}^{n} M_k(f)\Delta\alpha_k$ -$\sum_{k=1}^{n} m_k(f)\Delta\alpha_k < \frac{\varepsilon}{2} < \varepsilon$

(i.e.) U(p, f, α) -L(p,f, α) < ε

$\therefore$f$\in$R(α) on $[a, b]$

Theorem 3.6:

Each of the following conditions is sufficient for the existence of the Riemann integral $\int_a^b f(x)dx$:

a) f is continuous on $[a, b]$
b) f is of bounded variation on $[a, b]$

Proof:

Let f be a function defined on $[a, b]$

a) f is continuous on $[a, b] \Longrightarrow \int_a^b f(x)dx \; exists$

Given f is continuous on $[a, b]$

Let α (x) = x

$\Longrightarrow$ α is of bdd variation on $[a, b]$

$\therefore$ By theorem 3.5, we get

$\int_a^b f(x)dx$ exists

$(i.e.)$ $\int_a^b f(x)dx$ exists

$b)$ 'f' is of bounded variation on $[a, b] \Longrightarrow \int_a^b f(x)dx \; exists$

$\because$ $\alpha(x) = x$, α is continuous on $[a, b]$

$\therefore$ By theorem 3.5,

$$\int_a^b \alpha(x)df(x) \; exists$$

By theorem 2.9

$$\int_a^b f(x)d\alpha(x)\ exists$$

(i.e.) $\int_a^b f(x)dx\ exists$

Note:

By Theorem 2.9, a second sufficient condition can be obtained by interchanging 'f' & 'α' in the hypothesis.

(i.e.) If 'α' is a continuous on $[a,b]$ and if 'f' is of bounded variation on , then f∈R(α) on $[a,b]$

Necessary conditions for Existence of Riemann – stieltjes integrals

Theorem 3.7:

Assume that $\alpha \nearrow$on $[a,b]$ and let a<c<b. Assume further that both 'α' and 'f' are discontinuous from the right at x=c; that is, assume that there exists an ε >0 such that for every δ >0 there are values of x and y in the interval (c,c+δ) for which |f(x) – f(c)| ≥ ε and | α (y) – α (c)| ≥ ε. Then the integral $\int_a^b f(x)d\alpha(x)$ can not exists. The integral also fails to exist if 'α' and 'f' are discontinuous from the left at 'c'.

Proof:

Let p= {a=x_0, x_1,….,x_n =b}be a partition of $[a,b]$ containing 'c' as a point of subdivision.

Let on $\alpha \nearrow$on $[a,b]$

Let a<c<b

Given 'α' and 'f' are both discontinuous from the right at x=c.

(i.e.) there exists ε >0: $\forall\delta > 0$ x,y ∈ (c,c+δ) for which

|f(x) – f(c)| ≥ ε and | α (y) – α (c)| ≥ ε ………(1)

If the i^th subinterval has 'c' as its left end points, then

U(p, f, α) -L(p,f, α) = $\sum_{k=1}^{n}$ [M_k (f) - m_k(f)] $\Delta\, \alpha_k$

$= \sum_{k=1}^{n}$ [M_k (f) - m_k(f)] [α(x_i) – α(x_{i-1})]

$= \sum_{k=1}^{n}$ [M_k(f) - m_k(f)] [α(x_i) – α $(c) + \alpha(c) -$ α(x_{i-1})]

$\geq \sum_{k=1}^{n}$ [M_k(f) - m_k(f)] [α(x_i) – α(c)]

$\geq \sum_{k=1}^{n}$ [M_k(f) - m_k(f)] [α(x_i) – α(c)]

∴ U(p,f, α) -L(p,f, α) $\geq \sum_{k=1}^{n}$ [M_k(f) - m_k(f)] [α(x_i) – α(c)] ………(2)

Now,

M_i(f) – m_i(f) = sup f(x) – inf f(x) ,x∈[x_{i-1},x_i]

= sup f(x) – sup (-f(x))

= sup f(x) – sup f(y)

= sup (f(x) – f(y))

≥f(x) – f(c) : x,y∈ [x_{i-1}, x_i]

$\geq \varepsilon$

∴ M_i(f) – m_i(f) $\geq \varepsilon$ ………(3)

Now,

If ‘c’ is a common discontinuity from the right, we can assume that the point x_i is chosen so that

α(x_i) – α(c) $\geq \varepsilon$ ……….(4)

∴ $equation$ (2) ⟹ U(p, f, α) -L(p,f, α) $\geq \varepsilon. \varepsilon$

$= \varepsilon^2$

(i.e.) U(p, f, α) -L(p,f, α) $\geq \varepsilon^2$

∴ Riemann’s condition is not satisfied

∴ $\int_a^b f(x)d\alpha(x)$ Cannot exists.

Similarly, if ‘α’ and ‘f’ are discontinuous from the left at ‘c’ then we can prove that $\int_a^b f(x)d\alpha(x)$ doesn’t exists.

Mean – value Theorems for Riemann – stieltjes Integrals

Theorem 3.8: (First Mean – value Theorem for R.s Integral)

Assume that $\alpha \nearrow$on and let f∈R(α) on $[a,b]$. Let M & m denote, respectively, the sup and inf of the set {f(x):x∈ $[a,b]$}. Then there exists a real number 'c' satisfying m≤c≤M such that $\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) =$ c[$\alpha(b) - \alpha$ (a)]

In particular, if 'f' is continuous on $[a,b]$,then c=f(x_0) in

Proof:

Assume that on $\alpha \nearrow on\ [a,b]$

Let M= sup {f(x) : x∈ $[a,b]$} & m = inf {f(x) : x∈ $[a,b]$}

Let f∈R(α) on $[a,b]$

To prove: there exists c∈R satisfying m≤c≤M such that

$\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) =$ c[$\alpha(b) - \alpha$ (a)]

case (i): α(a)= α (b)

$\Rightarrow \alpha$ is constant on $[a,b]$

$$\Rightarrow \int_a^b f(x)d\alpha(x) = 0$$

Also $\alpha(b) - \alpha$ (a) = 0

$\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) =$ c[$\alpha(b) - \alpha$ (a)]

Case(ii): α (a)< α (b)

Let P= {a= x_0, x_1,….,x_n =b} ∈ $\mathcal{P}[a,b]$

Clearly, m≤f(t_k) ≤M ∀ t_k ∈ [x_{k-1},x_k]

$$\Rightarrow \sum_{k=1}^{n} m\Delta\alpha_k \le \sum_{k=1}^{n} f(t_k)\Delta\alpha_k \le \sum_{k=1}^{n} M\Delta\alpha_k$$

$$\Rightarrow m\sum_{k=1}^{n} \Delta\alpha_k \le S(p,f,\alpha) \le M\sum_{k=1}^{n} \Delta\alpha_k$$

$\Rightarrow$m $[\alpha(b) - \alpha$ (a)] ≤ S(p, f, α) ≤ M[$\alpha(b) - \alpha$ (a)]

$\Rightarrow$ m $[\alpha(b) - \alpha$ (a)] ≤ $\int_a^b f(x)d\alpha(x)$ ≤ M[$\alpha(b) - \alpha$ (a)]

$\Rightarrow$ m $\leq \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f(x)d\alpha(x) \leq$ M

$\Rightarrow$ m $\leq \frac{\int_a^b f(x)d\alpha(x)}{\int_a^b d\alpha(x)} \leq$ M

(i.e.) c= $\frac{\int_a^b f(x)d\alpha(x)}{\int_a^b d\alpha(x)}$

$\Rightarrow \int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) =$ c$[\alpha(b) - \alpha(a)]$ ....(1)

Now, Here 'f' is continuous on $[a, b]$ and m≤c≤M

By intermediate Theorem for continuous functions,

there exists $x_0 \in [a, b]$ such that $f(x_0) = c$

$\therefore$ *equation* (1) $\Rightarrow \int_a^b f(x)d\alpha(x) = f(x_0)\int_a^b d\alpha(x) = f(x_0)[\alpha(b) - \alpha(a)]$ for some $x_0 \in [a, b]$

Theorem 3.9: (Second Mean -value Theorem for R-S integral)

Assume that α is continuous and that f ↗ on $[a, b]$ then there exists a point x_0 in $[a, b]$ in such that $\int_a^b f(x)d\alpha(x) = f(a)\int_a^b d\alpha(x) + f(b)\int_a^b d\alpha(x)$

Proof:

Given 'α' is continuous and f ↗ on $[a, b]$

by first mean value Theorem 3.8, we get , there exists $x_0 \in [a, b]$ such that

$\int_a^b \alpha(x)df(x) = \alpha(x_0)[f(b) - f(a)]$ .... (1)

$$\int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

$\Rightarrow \int_a^b f(x)d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)df(x)$

$$= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)[f(b) - f(a)] \quad \textit{(by 1)}$$

$$= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)f(b) + \alpha(x_0)f(a)$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

$$\int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

The integral as a function of the interval

Note:

If f∈R(α) on $[a, b]$ and if 'α' is of bounded variation then the integral $\int_a^x f d\alpha$ exists x ∈ $[a, b]$

Theorem 3.10:

Let α be of bounded variation on $[a, b]$ and assume that f∈R(α) on [a,b] . Define F by equation F(x)= $\int_a^x f d\alpha$ if x ∈ $[a, b]$

Then we have

(i) F is of bounded variation on $[a, b]$

(ii) Every point of continuity of 'α' is also a point of continuity of F

(iii) If α ↗on $[a, b]$, the derivative F'(x) exists at each point x in (a , b) where α'(x) exists and where f is continuous. For such x, we have F'(x) = f(x) α'(x)

Proof:

Given, α be of bounded variation on[a,b] and f∈R(α) on [a,b]

Define 'F' by the equation F(x) =$\int_a^x f d\alpha$ if x ∈ $[a, b]$

(i)To Prove; F is of bounded variation on $[a, b]$

(i.e.) To prove: $\sum_{k=1}^{n} |\Delta F_k| \le M$ M>0

(i.e.)To prove: $\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \le M$ M>0

Assume that α ↗on $[a, b]$

Given, α is of bdd variation on [a,b]

$\sum_{k=1}^{n} |\Delta\, \alpha_{\,k}| \le N$ N>0

$\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \le N$ N>0 …….. (1)

Let m= inf{f(x) : x∈ [a,b]} & M= sup{f(x) : x∈ [a,b]}

$\because \alpha \nearrow$on $[a,b]$ and f∈R(α) on [a,b] by the first mean value theorem 3.8 we get,

There exists a real number 'c' satisfying $m \le c \le M$

such that $\int_a^b f(x)d\alpha(x) = c[\alpha(b) - \alpha(a)]$

If $x_{k-1} \neq x_k$, then $\int_{x_{k-1}}^{x_k} f(x)d\alpha(x) = c[\alpha(x_k) - \alpha(x_{k-1})]$

$$\Rightarrow \int_a^{x_k} f(x)d\alpha(x) - \int_a^{x_{k-1}} f(x)d\alpha(x) = c[\alpha(x_k) - \alpha(x_{k-1})]$$

$\Rightarrow F(x_k) - F(x_{k-1}) = c[\alpha(x_k) - \alpha(x_{k-1})]$

$\Rightarrow \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| = C \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})|$

$\le C.N$ (by equation (1))

= M where M=C.N

$$\therefore \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \le M$$

∴ F is of bounded variation on $[a,b]$

(ii) Every point of continuity of 'α' is also a point of continuity of F

Let 'α' be continuous at x_0

Let $\varepsilon > 0$ be given

To Prove: 'F' is continuous at x_0

∵ 'α' is continuous at x_0,

Given $\varepsilon>0$, there exists $\delta>0$ such that

$|x - x_0| < \delta \Rightarrow |\alpha(x) - \alpha(x_0)| < \varepsilon/c$ ....(2)

Now by theorem 3.8,

$\int_{x_0}^{x} f(x)d\alpha(x) = c[\alpha(x) - \alpha(x_0)]$

$\Rightarrow \int_a^x f(x)d\alpha(x) - \int_a^{x_0} f(x)d\alpha(x) = c[\alpha(x) - \alpha(x_0)]$

$\Longrightarrow$ F(x) – F(x_0)= $c[\alpha(x) - \alpha(x_0)]$ ………(3)

$\Longrightarrow$ |F(x) – F(x_0)|= $c|\alpha(x) - \alpha(x_0)|$

$< c.\ {}^{\varepsilon}/_{c}$ (by equation (2))

$\therefore |F(x) - F(x_0)| < \varepsilon$

$\therefore |x - x_0| < \delta$ $|F(x) - F(x_0)| < \varepsilon$

∴ F is continuous at x_0

∴ Every point of continuity of 'α' is also a point of continuity of F

(iii)Given: $\alpha \nearrow$ on $[a, b]$ & $\alpha'(x)$ exists & 'f' is continuous on [a , b]

To Prove: _F'(x) exists at each point x in (a , b)

Let $\varepsilon > 0$ be given

Let $x_0 \in (a,b)$

$\Longrightarrow \alpha'(x)$ exists

$\Longrightarrow \lim_{x\to x_0} \frac{\alpha(x)-\alpha(x_0)}{x-x_0}$ exists.

From equation (3) we have F(x) – F(x_0)= $c[\alpha(x) - \alpha(x_0)]$

$\Longrightarrow \frac{F(x) - F(x_0)}{x-x_0} = \frac{c[\alpha(x) - \alpha(x_0)]}{x-x_0}$

$\Longrightarrow \lim_{x\to x_0} \frac{F(x) - F(x_0)}{x-x_0} = c. \lim_{x\to x_0} \frac{[\alpha(x) - \alpha(x_0)]}{x-x_0}$

$\Longrightarrow F'(x_0) = c.\ \alpha'(x_0)$ ………. (4)

$\because \alpha'(x_0)$ exists, $\Longrightarrow$ $F'(x_0)$ also exists.

Here f is continuous on [a , b]

Then by intermediate value theorem,

There exists $x_0 \in [a, b]$ such that $f(x_0) = c$

$\therefore$ *From equation* (4) $\Longrightarrow F'(x_0) = c.\ \alpha'(x_0)$

$\Longrightarrow F'(x_0) = f(x_0)\alpha'(x_0)$

∵ x_0 is arbitrary we get

$F'(x) = f(x)\alpha'(x)$, $x\in (a,b)$

Theorem 3.11:

If f is continuous on [a,b] & $F(x)= \int_a^x f(x)dx$ then $F'(x) = f(x)$ on [a,b]

Proof:

Given, 'f' is continuous on [a,b] & $F(x)= \int_a^x f(x)dx$.

From part (iii) of theorem 3.10, we get

$F'(x) = f(x)\alpha'(x)$ ……. (1)

Let $\alpha(x) = x$

$\Longrightarrow \alpha'(x)= 1$ & $\alpha \nearrow$ on $[a, b]$

$\therefore (1) \Longrightarrow F'(x) = f(x).1$

$\Longrightarrow F'(x) = f(x)$ on [a,b]

Theorem 3.12:

[Conversation of Riemann integral of a Product of functions into R-S integral]

If $f\epsilon R$ & $g\epsilon R$ on [a,b], let $F(x)=\int_a^x f(t)dt$, $G(x)= \int_a^x g(t)dt$ if $x\epsilon[a, b]$. Then E & G are continuous functions of bounded variation on [a,b]. Also $f\epsilon R(G)$& $g\epsilon R(F)$ on [a,b], and we have $\int_a^b f(t)g(x)dx = \int_a^b f(t)dG(x) = \int_a^b g(x)dF(x)$

Proof:

Let $f\epsilon R$ & $g\epsilon R$ on [a,b]

Let $F(x)= \int_a^x f(t)dt$ & $\int_a^x g(t)dt$ if $x\epsilon[a, b]$

Let $\alpha(x) = x$

Assume that $\alpha \nearrow$ on [a,b]

Then by theorem 3.10 (i)& (ii) we get,

F & G are continuous functions of bounded variation on [a,b]

Also by Theorem 3.4, we get, $f \epsilon R(G)$& $g \epsilon R(F)$ on [a,b] &

$\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$

Second Fundamental Theorem of Integral Calculus

Theorem 3.13: [Second Fundamental Theorem of integral calculus]

Assume that $f \epsilon R$ on [a,b]. Let g be a function defined on [a,b] such that the derivative g' exists in (a,b) and has the value $g'(x) = f(x)\ \forall x \epsilon (a,b)$. At the end points assume that $g(a+)$ and $g(b-)$ exist and satisfy $g(a) - g(a+) = g(b) - g(b-)$. Then we have $\int_a^b f(x)dx = \int_a^b g^I(x)dx = g(b) - g(a)$.

Let P={a=$x_o, x_1, \dots, x_n$}$\epsilon\ p[a,b]$

∴ g is continuous on [a,b] & g' exists in (a,b) & by Mean-Value Theorem,

$g(x_k) - g(x_{k-1}) = g'(x_k).(x_k - x_{k-1})\ \forall\ t_k \epsilon (x_{k-1} - x_k)$ .... (1)

For every partition of [a, b] we can write

$g(b) - g(a) = \sum_{k=1}^{n}[g(x_k) - g(x_{k-1})]$

$= \sum_{k=1}^{n} g'(t_k).(x_k - x_{k-1})$ (by 1)

$= \sum_{k=1}^{n} g'(t_k).\Delta x_k$

$= \sum_{k=1}^{n} f(t_k).\Delta x_k$

∴ $g(b) - g(a) = \sum_{k=1}^{n} f(t_k).\Delta x_k$

Given $f \epsilon R \Rightarrow$ ⍰$A \epsilon \mathbb{R}$ ∋: ∀ $\varepsilon > 0$ p_ε of [a,b] ∋: ∀ p finer than p_ε & $t_k\ \epsilon[x_{k-1}, x_k]$, we have |S(P, f) – A|< ε where A= $\int_a^b f(x)dx$

$\Rightarrow |\sum_{k=1}^{n} f(t_k).\Delta x_k - \int_a^b f(x)dx| < \varepsilon$

$\Rightarrow |g(b) - g(a) - \int_a^b f(x)dx| < \varepsilon$

$\Rightarrow \left| \int_a^b f(x)dx - (g(b) - g(a)) \right| < \varepsilon$

$\Rightarrow \int_a^b f(x)dx = g(b) - g(a)$

$\Rightarrow \int_a^b f(x)dx = \int_a^b g'(x)dx = g(b) - g(a)$

Theorem 3.14:

Assume $f\epsilon R$ on [a,b]. let α be a function which is continuous on [a, b] and whose derivative α' is Riemann integrable on [a, b]. Then the following integrals exist and are equal $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$.

Proof:

By the Second Fundamental Theorem we get,

$\alpha(x) - \alpha(a) = \int_a^b \alpha'(t)dt \ \forall \ x\epsilon[a,b]$ …….. (1)

By Theorem 3.12 we get,

$\int_a^b f(x)g(x)dx = \int_a^b f(x)dGx$ …….. (2)

Where G(x)= $\int_a^x g(t)dt$

Let $g = \alpha'$

Then G$(x) = \int_a^b \alpha'(t)dt$

$\Rightarrow G(x) = \alpha(x) - \alpha(a)$

$\Rightarrow dG(x) = d\alpha(x) - d\alpha(a)$

$\Rightarrow dG(x) = d\alpha(x) - 0$

$\Rightarrow dG(x) = d\alpha(x)$

$\therefore$ *equation* (2) becomes $\int_a^b f(x)g(x)dx = \int_a^b f(x)dGx$

$\int_a^b f(x)\alpha'(x)dx = \int_a^b f(x)d\alpha(x)$

Change of Variance

Theorem 3.15: [(Change of Variable in a Riemann integral)]

Assume that g has a continuous derivative g' on the interval [c, d]. Let f be continuous on $g([c,d])$ and define F by the equation $F(x) = \int_{g(c)}^{x} f(t)dt$ if $x\epsilon g([c,d])$. Then for each x in [c,d] the integral $\int_{c}^{x}[g(t)]\, g'(t)dt$ exists and has the value $F[g(x)]$ in particular. We have $\int_{g(c)}^{g(d)} f(x)dx = \int_{c}^{d} f[g(t)]\, g'(t)dt.$

Proof:

Assume that g has a continuous derivative g' on [c, d]

Let f be continuous on $g([c,d])$

Define $F(x) = \int_{g(c)}^{x} f(t)dt$ if $x\epsilon g([c,d])$ ………..(1)

To Prove: $\int_{c}^{x} f[g(t)]\, g^{I}(t)dt = F[g(x)]$ exists

Here f is continuous on $g([c,d])$

∴ By Theorem 3.6, we get $\int_{g(c)}^{g(d)} f(x)dx$ exists

Also f is continuous on $g([c,d])$ &

g is continuous on [c, d]

⇒ fog is continuous on [c, d]

& also we have g' is continuous on [c, d]

⇒ $(fog).g'$ is continuous on [c, d]

By Theorem 3.6, we get

$\int_{c}^{d}(fog)(t)\, g'(t)dt$ exists

(i.e.) $\int_{c}^{d} f[g(t)]\, g'(t)dt$ exists

Define G on [c, d] as follows:

G(x)= $\int_c^d f[g(t)]\, g'(t)dt$ ....(2)

To Prove: G(x)=$F[g(x)]$

Now, By first Fundamental Theorem of Integral Calculus

$\therefore$ *equation* (2) $\Rightarrow G'$(x)= $F[g(x)]g'(x)$ ....(3)

Now, By the chain rule of differentiation, we get

$(F[g(x)])' = F'\big(g(x)\big).\, g'(x)$

$= f\big(g(x)\big).\, g'(x)$ $\quad [\;\therefore F'(x) = f(x)]$

$\therefore\; (F\big(g(x)\big))' = f\big(g(x)\big)g'(x)$ .... (4)

$\therefore\; [G(x) - F\big(g(x)\big)]' = G'(x) - \big[F\big(g(x)\big)\big]'$

$= f\big(g(x)\big).\, g'(x) - f(g(x))g'(x)$ (by equation (3) & (4))

$\therefore\; [G(x) - F\big(g(x)\big)]' = 0$

$\Rightarrow\; G(x) - F\big(g(x)\big)'$ is a constant

Sup x= c,

Then G(c)= $\int_c^d f[g(t)]\, g'(t)dt = 0$ (by equation (2))

& $F(g(c)) = \int_{g(c)}^{g(c)} f(t)dt = 0$ (by equation (1))

$\therefore\; G(c) = F\big(g(c)\big) = 0$

$\therefore\; G(x) - F\big(g(x)\big) = 0\; \forall\; x\epsilon[c, d]$

$\Rightarrow G(x) = F[g(x)]\; \forall\; x\epsilon[c, d]$

In particular, if x =d, then

G(d)$-F(g(d)) = 0$

$\Rightarrow G(d)$= $F\big(g(d)\big)$

(i.e.)., $\int_c^d f[g(t)]g'(x)dx = \int_{g(c)}^{g(d)} f(x)dx$ (by 1&2)

Note: [General Theorem on change of variable in a Riemann integral]

Assume that h ϵR on [c, d] and if $x\epsilon[c,d]$, Let $g(x) = \int_a^x h(t)dt$, where 'a' is a fixed point in [c,d]. Then if $f\epsilon R$ on $g([c,d])$, then the integral $\int_c^d f[g(t)]h(t)dt$ exists and we have $\int_{g(c)}^{g(d)} f(x)dx = \int_c^d f[g(t)]h(t)dt.$

Second Mean- Value Theorem for Riemann Integrals

Theorem 3.16:

Let g be continuous and assume that $f \nearrow$ on [a,b]. Let A and B be two real numbers satisfying the inequalities A≤ $f(a+)$ and B≥ $f(b-)$. Then there exists a point x_o in [a.b] such that

(i) $\int_a^b f(x)g(x)dx = A\int_a^{x_o} g(x)dx + B\int_{x_o}^b g(x)dx$. In particular, if $f(x) \geq 0$ $\forall x\epsilon[a,b]$, we have (ii) $\int_a^b f(x)g(x)dx = B\int_{x_o}^b g(x)dx$ where x_o $\epsilon[a,b]$ part (ii) is known as Bonnet's Theorem.

Proof:

Let 'g' be continuous & $f \nearrow$ on [a,b]

Let A & B be two real numbers ∋: A≤ $f(a+)$ & B≥ $f(b-)$.

(i) Let $\alpha(x) = \int_a^x g(t)dt.$

$\Rightarrow \alpha'(x) = g(x)$

Here 'α' is continuous & $f \nearrow$ on [a,b]

Then by second Mean- Value Theorem for R-S integral Theorem7.31, we get

$$\int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_o} d\alpha(x) + f(b)\int_{x_o}^b d\alpha(x)$$

$$\Rightarrow \int_a^b f(x)\alpha'(x)dx = f(a)\int_a^{x_o} \alpha'(x)dx + f(b)\int_{x_o}^b \alpha'(x)dx$$

$$\Rightarrow \int_a^b f(x)g(x)dx = f(a)\int_a^{x_o} g(x)dx + f(b)\int_{x_o}^b g(x)dx.$$

$$\Rightarrow \int_a^b f(x)g(x)dx = A\int_a^{x_o} g(x)dx + B\int_{x_o}^b g(x)dx. \quad \text{..........(1)}$$

Where A=$f(a)$ & $B = f(b)$

If A & B are any two real numbers satisfying A$\leq f(a+)$ & B$\geq f(b+)$, then we can redefine the end points a & b to have A=$f(a)$ & $B = f(b)$.

(ii)Given $f \nearrow$ on [a,b]

⇒Modified 'f' is still increasing on [a, b]

Also we know that changing the value of 'f' at a finite number of points does not affect the values of a Riemann integral.

Take A=a, we get,

From equation (1) ⇒ $\int_a^b f(x)g(x)dx = B\int_{x_o}^b g(x)dx.$

Riemann – satisfies integrals Depending on a Parameters

Theorem 3.17:

Let f be continuous at each point (x,y) of a rectangle Q= {(x, y) : a$\leq x \leq b, c \leq y \leq d\}$. Assume that α is of bounded variation on [a,b] and Let F be the function defined on [c,d], by the equation F(y)= $\int_a^b f(x,y)d\alpha\,(x)$. Then F is continuous on [c, d]. In other words, if $y_o \epsilon [c,d]$. We have $\lim_{y\to y_o} \int_a^b f(x,y)d\alpha\, x = \int_a^b \lim_{y\to y_o} f(x,y)d\alpha\, x = \int_a^b f(x,y_o)d\alpha\, x.$

Proof:

Given Q= {(x, y) : a$\leq x \leq b, c \leq y \leq d\}$

Let F(y)= $\int_a^b f(x,y)d\alpha\,(x)$ …………(1)

To Prove: F is continuous on [c, d]

Assume that $\alpha \nearrow$ on [a,b]

∴ Q is a compact set, f is uniformly continuous on Q

⇒ Given $\varepsilon > 0, \ni \delta > 0 \ni: |z - z'| < \delta \Rightarrow |f(z) - f(z)'| < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$

where z= (x,y) & z'= $(x', y')\epsilon\, Q$.

$\Rightarrow |z-z'| < \delta \Rightarrow |f(x,y) - f(x',y')| < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ ……… (2)

If $|y-y'| < \delta$ we have

$|f(y) - f(y)'| = |\int_a^b f(x,y)d\alpha\,(x) - \int_a^b f(x,y')d\alpha\,(x)|$ (by 1)

$= |\int_a^b [f(x,y) - \int_a^b f(x,y')]d\alpha\,(x)|$

$\leq \int_a^b |f(x,y) - f(x,y')|d\alpha\,(x)$

$< \frac{\varepsilon}{\alpha(b)-\alpha(a)} \int_a^b d\alpha\,(x)$ (by equation (2))

$= \frac{\varepsilon}{\alpha(b)-\alpha(a)} [\alpha(b) - \alpha(a)]$

$|f(y) - f(y)'| < \varepsilon$

(i.e.).,Given $\varepsilon > 0$, $\exists\ \delta > 0 \ni: |y-y'| < \delta \Rightarrow |f(y) - f(y)'| < \varepsilon$

Hence F is continuous on [c, d]

(i.e.)., If $y_o \epsilon [c,d]$, then

$\Rightarrow \lim_{y\to y_o} \int_a^b f(x,y)d\alpha x = \int_a^b f(x,y_o)d\alpha x$

$\Rightarrow \int_a^b \lim_{y\to y_o} f(x,y)d\alpha x = \int_a^b f(x,y_o)d\alpha x$

$\lim_{y\to y_o} \int_a^b f(x,y)d\alpha x = \int_a^b \lim_{y\to y_o} f(x,y)d\alpha x = \int_a^b f(x,y_o)d\alpha x$

Theorem 3.18:

If f is continuous on the rectangle [a, b] × [c,d] and if $g\epsilon R$ on [a,b], then the function F defined by the equation F(y)= $\int_a^b g(x)f(x,y)dx$, is continuous on [c,d]. That is if $y_o \epsilon [c,d]$, we have $\lim_{y\to y_o} \int_a^b g(x)f(x,y)dx = \int_a^b g(x)f(x,y_o)dx$

Proof:

Given F(y)= $\int_a^b g(x)f(x,y)dx$ ………(1)

To prove: F is continuous on [c, d]

Let $G(x) = \int_a^b g(x)dx$

By Theorem 3.12 we get,

$\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x)$

(i.e.)., $\int_a^b f(x,y)g(x)dx = \int_a^b f(x,y)dG(x)$

$\Rightarrow F(y) = \int_a^b f(x,y)dG(x)$

By Theorem 3.17, we get F is continuous on [c, d]

(i.e.) If $y_o \epsilon [c,d]$, $\lim_{y \to y_o} F(y)$= F(y_o)

$\Rightarrow \lim_{y \to y_o} \int_a^b g(x)f(x,y)dx = \int_a^b g(x)f(x,y_o)dx$

Unit IV

Infinite Series and infinite Products - Double sequences - Double series -Rearrangement Theorem for double series - A sufficient condition for equality of iterated series - Multiplication of series – Cesaro summability - Infinite products.

Power series - Multiplication of power series - The Taylor's series generated by a function - Bernstein's Theorem– Abel's limit Theorem– Tauber's theorem.

Infinite series and Infinite Products

Double Sequences:

Definition 4.1:

A function f whose domain is $Z^+ \times Z^+$ is called a double sequences.

Definition 4.2:

If a∈C, we write $\lim_{p,q\to\infty} f(p,q)$ = a and we say that the double sequence f converges to 'a', provided that the following condition is satisfied:

For all ε> 0 there exist N such that |f(p, q)-a|< ε whenever p, q> N

Note:

$\lim_{p,q\to\infty} f(p,q)$ Is call a double limit.

$\lim_{p\to\infty} \lim_{q\to\infty} f(p,q)$Is called an iterated limit.

Theorem 4.3:

Assume that $\lim_{p,q\to\infty} f(p,q)$ = a. For each fixed p. Assume that the $\lim_{q\to\infty} f(p,q)$ exists. Then the limit $\lim_{p\to\infty} (\lim_{q\to\infty} f(p,q))$ also exists and has the value 'a'.

Proof:

$\lim_{p,q\to\infty} f(p,q)$ = a

$\Rightarrow$ given $\varepsilon>0$, there exist N_1 such that

$|f(p, q)-a|<\varepsilon/2$ whenever $p, q>N_1$ ………… (1)

Given $\lim_{q\to\infty} f(p,q)$ exists

Let $F(p) = \lim_{q\to\infty} f(p,q)$

For each p there exist N_2 such that,

$|F(p) - f(p, q)| < \varepsilon/2$ whenever $q > N_2$

For each $p > N_1$ choose N_2 and then choose a fixed q greater than both N_1 & N_2

$\therefore |F(p) - a| = |F(p) - f(p, q) + f(p, q) - a|$

$= |F(p)-f(p, q)| + |f(p, q)-a|$

$= \varepsilon/2 + \varepsilon/2 = \varepsilon$

$|F(P)-a| < \varepsilon$

$\lim_{p\to\infty} F(p) = a$

$\lim_{p\to\infty} (\lim_{q\to\infty} f(p,q)) = a$

Hence the existence of the double limit $\lim_{p,q\to\infty}$ f(p,q) and the limit $\lim_{q\to\infty}$ f(p, q) implies the existence of the iterated limit $\lim_{p\to\infty} (\lim_{q\to\infty} f(p,q))$.

Note:

The converse of the above Theorem is not true

Let $f(p, q) = \frac{pq}{p^2+q^2}$ (p=1,2,…… ; q=1,2,……..)

Then $\lim_{q\to\infty} f(p,q) = \lim_{q\to\infty} \frac{pq}{p^2+q^2} = \lim_{q\to\infty} \frac{p}{q(\frac{p^2}{q^2}+1)} = 0$

$\lim_{q\to\infty} f(p,q) = 0$

But when p=q, $f(p, q) = \frac{p^2}{p^2+p^2} = \frac{p^2}{2p^2} = 1/2$

& when p=2q, f(p, q) = $\frac{2q^2}{4q^2+q^2}$ = $\frac{2q^2}{5q^2}$ = 2/5

(i.e.), The double limit cannot exist in this case.

Double Series

Definition 4.4:

Let f be a double sequence and let S be the double sequence defined by the equation

S(p, q) = $\sum_{m=1}^{p}\sum_{n=1}^{q}(f(m,n)$

The pair (f, s) is called a double series and is denoted by

$\sum_{m,n}(f(m,n)$ or $\sum$f(m, n)

The double series is said to be converge to the sum 'a' if

$\lim_{p,q\to\infty} S(p,q)$=a

Note:

- Each number f (m, n) is called a term of the double series.
- Each S (p, q) is a partial sum of the double series.
- A double series of positive terms converges if and only if the set of partial term is bounded. We say $\sum$f (m, n) converges absolutely if

  $\sum$|f (m, n) |converges

- A double series converges absolutely implies A double series converges.

Rearrangement Theorem for Double series

Definition 4.5:

Let f be a double sequence and let 'g' be a one to one function defined on Z^+ with range Z^+ x Z^+. Let G be the Sequence defined by G(n) = f[g(n)] if n$\in Z^+$

Then g is said to be an arrangement of the double sequence f into the sequence G

Theorem 4.6:

Let ∑f (m, n) be a given double series and let 'g' be a arrangement of the double sequence f into the Sequence G. Then

(a) ∑G(n) converges absolutely if and only if ∑ f(m, n) converges absolutely.
Assume that ∑ f (m, n) does converges absolutely with sum S, we have further:

(b) $\sum_{n=1}^{\infty} G(n)$=S
(c) $\sum_{n=1}^{\infty} f(m,n)$ & $\sum_{m=1}^{\infty} f(m,n)$ both converges absolutely
(d) If $A_n = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{n=1}^{\infty} f(m,n)$ both series $\sum A_n$ & $\sum B_n$ converges absolutely and both have sum S.
(i.e.),$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S$

Proof:

Let ∑f (m, n) be a given double series

Let 'g' be an arrangement of the double sequence f into the sequence G

G(n) = f[g(n)] if $n \in Z^+$

a) Let $T_k = |G(1)|+|G(2)|+\dots\dots+|G(k)|$
Let $S(p, q) = \sum_{m=1}^{p} \sum_{n=1}^{q} |f(m,n)|$

Then, For each k, there exist a pair (p, q) such that $T_k \le S(p, q)$

Conversely,

For each pair (p, q), there exist an integer 'r' such that $S(p, q) \le T_r$

∑|G(n)| has bounded partial sums if and only if ∑|f(m, n)| has bounded partial sums.

∑|G(n)| converges if and only if ∑|f (m, n) converges

(i.e.) ∑|G(n)| converges absolutely if and only if ∑|f (m, n) converges absolutely

b) Assume that ∑f (m, n) converges absolutely
(i.e.), ∑| f (m, n)| converges with sum S

Let 'g' be an arrangement of f (m, n) into G

To prove:$\sum_{n=1}^{\infty}$ G(n) = S

First, we shall show that the sum of the series $\sum G(n)$ is independent of the function g used to construct G from f.

Let h be an another arrangement of the double sequence f(m, n) into sequence H

We have, G(n) = f[g(n)] & h(n) = f[h(n)]

Now,

H(n) = f(h(n))

$H(h^{-1}(n)) = f(h(h^{-1}(n)))$

$H(h^{-1}(n) = f(n)$

$H(h^{-1}(g(n))) = f(g(n))$

G(n) = f[g(n)] becomes $G(n) = H(h_{-1}(g(n)))$

G(n) = H(k(n) where $k(n) = h_{-1}(g(n))$

Now,

We have g: $Z^+ \times Z^+$ & h-1: $Z^+ \times Z^+ \rightarrow Z^+$

$h^{-1} \circ g: Z^+ \times Z^+$

(i.e.) k is a 1-1 mapping of Z^+ onto Z^+

$\sum H(n)$ is a rearrangement of $\sum G(n)$

$\sum H(n)$ & $\sum G(n)$ has the same sum

To show that $S = S'$

Let $T = \lim_{p,q\to\infty} S(p,q)$

Given ε > 0, choose N so that

$0 \leq |T - S(p, q)| < \varepsilon/2$ whenever p, q>N ……….(1)

Let $t_k = \sum_{n=1}^{k} G(n($, $S(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} f(m,n)$

Choose M so that t_M includes all terms f(m, n) with $1 \leq m \leq N+1$ & $1 \leq n \leq N+1$

Then t_M – S (N+1, N+1) is a sum of terms f(m, n) with m>N or n>N

If n ≥ M, we have

$|t_M - S(N+1, N+1)| \leq T - S(N+1, N+1) < \varepsilon/2$ (by equation (1))

Similarly

$|S - S(N+1, N+1)| \leq T - S(N+1,N+1) < \varepsilon/2$

Given ε>0, we can find M so that $|t_n - S| < \varepsilon$ whenever n ≥ M

$\Rightarrow \lim_{n\to\infty} t_n = S$

But we have $\lim_{n\to\infty} t_n = S'$

$\therefore S = S'$

Hence $\sum_{n=1}^{\infty} G(n) = S$

c) Each series $\sum_{n=1}^{\infty} f(m,n)$ & $\sum_{m=1}^{\infty} f(m,n)$ are the sub series of ∑G(n).
We have Σ G(n) converges absolutely

→ Sub series $\sum_{n=1}^{\infty} f(m, n)$ & $\sum_{m=1}^{\infty} f(m, n)$ of ∑G(n) are converges absolutely

(d) Given: $A_m = \sum_{n=1}^{\infty} f(m,n)$ & $B_n = \sum_{n=1}^{\infty} f(m,n)$

To prove: $\sum A_m$ & $\sum B_n$ converges absolutely and both have sum S.

we conclude that

$\sum A_m$ converges absolutely & have sum S

$\sum b_n$ converges absolutely & have sum S

Note:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) \neq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)$$

Both the series are "Iterated series"

For example,

Sup $f(m, n)=\begin{cases} 1 & if\ m = n+1, 1,2,\ldots \\ -1 & if\ m = n-1, n = 1,2\ldots \\ 0 & otherwise \end{cases}$

Then $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) = -1$ & $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)=1$

Definition 4.7:

Let f be a function whose domain is Z^+ and whose range is an infinite subset of Z^+, and assume that f is 1-1 on Z^+. Let $\sum a_n$ & $\sum b_n$ be two series such that $b_n = a_{f(n)}$ if $n \in Z^+$. Then $\sum b_n$ is said to be a subseries of $\sum a_n$

Theorem 4.8:

If $\sum a_n$ converges absolutely, every subseries b_n also converges absolutely. Moreover, we have

$|\sum_{n=1}^{\infty} b_n| \leq \sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} |a_n|$

Proof:

Given 'n', let N be the largest integer in the set {f(1),f(2),…..,f(n)}

Then $|\sum_{k=1}^{n} b_n| \leq \sum_{k=1}^{n} |b_k| \leq \sum_{k=1}^{N} |a_k| \leq \sum_{k=1}^{n} |a_k|$

$\therefore \sum_{k=1}^{n} |b_k| \leq \sum_{k=1}^{n} |a_k|$

$\Rightarrow \sum b_n$ converges absolutely

Theorem 4.9:

Let $\{f_1, f_2 \ldots\}$ be a countable collection of functions, each defined on Z^+, having the following properties

(a) each f_n is 1-1 on Z^+

(b) The range $f_n(Z^+)$ is a subset Q_n of Z^+

(c) $\{Q_1\ Q_2, \ldots\ldots\}$ is a collection of disjoint sets whose union is Z^+

Let $\sum a_n$ be an absolutely convergent series and define

$b_k(n)=a_{f_k(n)}$ if $n \in Z^+$, $k \in Z^+$

Then,

(i) For each k, $\sum_{n=1}^{\infty} b_k(n)$ is an absolutely convergent Subseries of $\sum a_n$

(ii) If $S_k = \sum_{n=1}^{\infty} b_k(n)$, the series $\sum_{k=1}^{\infty} S_k$ converges absolutely and has the same sum as $\sum_{k=1}^{\infty} a_k$

Proof:

Given $\sum a_n$ converges absolutely.

The subseries $\sum_{n=1}^{\infty} b_k(n)$ also converges absolutely.

To prove, $\sum_{k=1}^{\infty} S_k$ converges absolutely & has sum $\sum_{k=1}^{\infty} a_k$

Let $t_k = |S_1|+|S_2|+\ldots\ldots+|S_k|$

Then

$$t_k \leq \sum_{n=1}^{\infty} |b(n)\}+\ldots\ldots+\sum_{n=1}^{\infty} |b_k(n)|$$

$$= \sum_{n=1}^{\infty} (|b_1(n)|+b_2(n)+\ldots\ldots|b_k(n)|)$$

$$= \sum_{n=1}^{\infty} (|a_{f1(n)}|+\ldots\ldots..+|a_{fk(n)}|)$$

$$\leq \sum_{n=1}^{\infty} |a_n|$$

$$\therefore t_k \leq \sum_{n=1}^{\infty} |a_n|$$

$\therefore \sum |S_k|$ has bounded partial sums

$\therefore \sum S_k$ converges

(i.e.) $\sum S_k$ converges absolutely

Now, to prove the sum: $\sum S_k$ is $\sum a_k$

Let $\varepsilon > 0$ be given.

Choose N so that $n \geq N \Rightarrow \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} |a_k| < \varepsilon/2$ ....(1)

Choose enough functions $f_1, f_2, \ldots..f_r$ so that each term $a_1, a_2, \ldots, a_N$ will appear somewhere in the sum

$\sum_{n=1}^{\infty} a_{f1(n)}+\ldots..+\sum_{n=1}^{\infty} a_{fr(n)}$

The number r depends on N & hence on ε If n>r & n > N. we have

$|S_1+S_2+\ldots+S_n - \sum_{k=1}^{n} a_k| \leq |a_{N+1}|+|a_{N+2}|+\ldots<\varepsilon/2$

$|S_1+S_2+\ldots+S_n - \sum_{k=1}^{n} a_k| <\varepsilon/2$ ……….(2)

Now,

$|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} |a_k| < \varepsilon/2$

$|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k| < \varepsilon/2$ ………..(3)

From equation (2) & (3) we get

$|S_1+S_2+\ldots+S_n - \sum_{k=1}^{n} a_k| <\varepsilon$ if n>r, n>N

∴their sum $\sum S_k$ is $\sum a_k$

The subseries also has the same as the series

A sufficient condition for equality of Iterated Series

Theorem 4.10:

Let f be a complex-valued double sequence. Assume that $\sum_{n=1}^{\infty}$ f(m,n) converges absolutely for each fixed m and that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$ converges. Then,

a) The double series $\sum_{m,n}$ f(m,n) converges absolutely

b) The series $\sum_{m=1}^{\infty}$ f(m,n) converges absolutely for each `n'.

c) Both iterated series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n)$ converges absolutely and we have $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{m,n} f(m,n)$

Proof:

Let f be a complex-valued double sequence

Assume that $\sum_{n=1}^{\infty}$ f(m,n) converges absolutely for all fixed m

& $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$ converges.

$\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty}$ f(m,n) converges absolutely

Let 'g' be an arrangement of the double sequence 'f' in to the sequence G.

∴G(n) = f[g(n)] if $n \in Z^+$

All the partial sums of $\sum|G(n)|$ are bounded by $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty}$ f(m,n)

$\therefore \sum G(n)$ converges absolutely

$\Rightarrow \sum_{m,n}$ f(m, n) converges absolutely (by theorem 4.6 (a))

By Theorem 4.6 (c),

$\sum_{m=1}^{\infty}$ f(m, n) converges absolutely $\forall$ fixed 'n'.

By Theorem 4.6 (d),

$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty}$ f(m,n) & $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty}$ f(m,n) converges absolutely and $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty}$ f(m,n) = $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{m,n}$ f(m,n)

Theorem 4.11:

Let $\sum a_m$ and $\sum b_n$ be two absolutely convergent series with sums A & B, respectively. Let f be the double sequence defined by the equation

$f(m, n) = a_m b_n$ if $(m, n) \in Z^+ \times Z^+$

Then f(m, n) converges absolutely and has the sum AB

Proof:

Let $\sum a_m$ & $\sum b_n$ be two absolutely converges series with sums A & B, respectively.

Let f be the double sequence by $f(m, n) = a_m b_n$ if $(m, n) \in Z^+ \times Z^+$ TP: $\sum_{m,n}$ f(m,n) converges absolutely & has sum AB

Now, $\sum_{m=1}^{\infty} |a_m| \sum_{n=1}^{\infty} |b_n| = \sum_{m=1}^{\infty} (|a_m| \sum_{n=1}^{\infty} |b_n|)$

$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m||b_n|$

$\sum_{m=1}^{\infty} |a_m| \sum_{n=1}^{\infty} |b_n| = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m b_n|$

$\therefore$ The double series $\sum_{m,n} a_m b_n$ converges absolutely & has the sum AB (by theorem 4.10)

(i.e.),$\sum_{m,n}$ f(m,n) converges absolutely & has sum AB

Multiplication of Series:

Definition 4.12:

Given two series $\sum_{n=0}^{\infty}$ a_n and $\sum_{n=0}^{\infty}$ b_n define

$C_n = \sum_{k=0}^{\infty}$ a_k b_{n-k} if n=1,2,….

The series $\sum_{n=0}^{\infty}$ C_n is called the Cauchy product of $\sum a_n$ & $\sum b_n$

Note:

- $\sum a_n$ & $\sum b_n$ converges absolutely => $\sum C_n$ converges and $\sum C_n = (\sum a_n)(\sum b_n)$
- This equation may fall to hold if $\sum a_n$ & $\sum b_n$ conditionally convergent
- (i.e.), $\sum a_n$ & $\sum b_n$ conditionally convergent not implies C_n converges
- If either $\sum a_0$ (or) $\sum b_n$ converges absolutely implies c_n converges

Theorem 4.13: [Mertens Theorem]

Assume that $\sum_{n=0}^{\infty}$ a_n converges absolutely and has sum A, and suppose $\sum_{n=0}^{\infty}$ b_n converges with sum B. Then the Cauchy product of these two series converges and has sum AB.

Proof:

Given $\sum_{n=0}^{\infty}$ a_n converges absolutely and has sum A & $\sum_{n=0}^{\infty}$ b_n converges with sum B

Let the Cauchy product of $\sum a_n$ & $\sum b_n$ be $\sum_{n=0}^{\infty}$ C_n and define

$C_n = \sum_{n=0}^{\infty}$ a_k b_{n-k} if n=1,2,….

Define $A_n = \sum_{k=0}^{n}$ a_k & $B_n = \sum_{k=0}^{n}$ b_k & $C_n = \sum_{k=0}^{\infty}$ C_k ………….(2)

Let $d_n = B - B_n$ & $e_n = \sum_{k=0}^{n}$ a_k d_{n-k} ……………….(3)

Define $f_n(k) = \begin{cases} a_k b_{n-k} & if\ n \geq k \\ 0 & if\ n < k \end{cases}$

Then

$C_p = \sum_{n=0}^{p}$ $\sum_{k=0}^{n}$ a_k b_{n-k}

$= \sum_{n=0}^{p} b_{n-k} \sum_{k=0}^{n} a_k$

$= \sum_{k=0}^{p} a_k \sum_{k=0}^{p} b_{n-k}$

$= \sum_{k=0}^{p} a_k \sum_{m=0}^{p-k} b_m$

$= \sum_{k=0}^{p} a_k B_{p-k}$ (by 2)

$=\sum_{k=0}^{p} a_k (B-d_{p-k})$ (by 3)

$= \sum_{k=0}^{p} a_k B - \sum_{k=0}^{p} a_k d_{p-k}$

$C_p = A_p B - e_p$ (by equation (2) & (3)) ……… (4)

It is sufficient to show that $e_p \to 0$ as $p\to\infty$

(3) $\Rightarrow d_n=B-B_n = \sum_{n=1}^{\infty} b_n - \sum_{k=0}^{n} b_k$

$\therefore \{d_n\}\to 0$ ($\because \sum b_n$ converges)

$\Rightarrow \{d_n\}$ is bounded

$\Rightarrow$Choose M>0 so that $|d_n|\leq M \ \forall n$ ………… (5)

Let $K = \sum_{n=0}^{\infty} |a_n|$

Now, $\{d_n\}\to 0$ & $\sum|a_n|$ converges

$\Rightarrow$Given $\varepsilon > 0$, Choose N so that

$n>N \Rightarrow |d_n|< \varepsilon/2K$

& $n > N \Rightarrow \sum_{n=N+1}^{\infty} |a_n|< \varepsilon/2M$

For p>2N, we can write

$3 \Rightarrow |e_p| = |\sum_{k=0}^{p} a_k d_{p-k}|$

$= |\sum_{k=0}^{N} a_k d_{p-k} + \sum_{k=N+1}^{p} a_k d_{p-k}|$

$\leq |\sum_{k=0}^{N} a_k d_{p-k} | +| \sum_{k=N+1}^{p} a_k d_{p-k}|$

$\leq \varepsilon/2K \sum_{k=0}^{N} |a_k| + M \sum_{k=N+1}^{p} |a_k|$ (by equation (5) & (6))

$< \frac{\varepsilon}{2K}.K + M \frac{\varepsilon}{2M}$

$<\varepsilon/2 + \varepsilon/2$

$= \varepsilon$

$\therefore |e_p| < \varepsilon$

$\therefore e_p \to 0$ as $p \to \infty$

From equation (4) $\Rightarrow C_p = A_p B - e_p$

$\Rightarrow C_p = A_p B$

$\Rightarrow C_p \to AB$ as $p \to \infty$

(i.e.), the Cauchy product of two series converges and as the sum AB

Definition 4.14: [Dirichlet Product or Dirichlet Convolution]

Given two series Σa_n & $\sum b_n$, define $C_n = \sum_{d/n} a_d b_{n/d}$ (n=1,2….) Where $\sum_{d/n}$ means a sum extended over all positive divisors of 'n' (including i and n). This product $\sum C_n$ is known as Dirichlet product.

Note:

Take $a_0=b_0=0$ in the Cauchy product $\sum C_n$, where $C_n = \sum_{k=0}^{n} a_k b_{n-k}$

(n= 0,1,2, …….)

we get the dirichlet product $\sum C_n$

& $C_n = \sum_{d/n} a_d b_{n/d}$

For example,

$C_6 = a_1 b_6 + a_2 b_3 + a_3 b_2 + a_6 b_1$

$C_7 = a_1 b_7 + a_7 b_1$

Definition 4.15:

A series of the form $\sum_{n=1}^{\infty} a_n/n^s$ is called a Dirichlet series.

Definition 4.16:

Given two absolutely convergent Dirichlet series, say $\sum_{n=1}^{\infty} a_n/n^s$ and $\sum_{n=1}^{\infty} b_n/n^s$, having sums A(S) & B(S), respectively Then$\sum_{n=1}^{\infty} C_n/n^s$ = A(S)B(S)

where $C_n = \sum_{d/n} a_d\, b_{n/d}$ is the product of two dirichlet series.

Cesaro Summability:

Definition 4.17:

Let S_n denote the nth partial sum of the series $\sum a_n$ and let $\{\sigma_n\}$ be the sequence of arithmetic means defined by $\sigma_n = \frac{S_1+S_2+\ldots+S_n}{n}$, if n=1,2,... The series $\sum a_n$ is said to be Cesaro Summable (or) (C,1) summable If $\{\sigma_n\}$ converges. $\lim_{n\to\infty} \sigma_n = S$, then S is called the Cesaro Sum (or) (C,1) sum of $\sum a_n$, and we write $\sum a_n = S$, (c,1)

Example 4.18:

Let an = Z-1, , |Z|=1, Z≠1.

Then $S_n = a_1+a_2+\ldots+a_n = 1+Z+Z^2+\ldots+Z^{n-1} = \frac{1-Z^n}{1-Z}$

$\therefore S_n = \frac{1-Z^n}{1-Z} = \frac{1}{1-Z} - \frac{Z^n}{1-Z}$ & $\sigma_n = \frac{S_1+S_2+\ldots+S_n}{n}$

$$= \frac{1}{n}\left[\left(\frac{1}{1-Z} - \frac{Z}{1-Z}\right) + \left(\frac{1}{1-Z} - \frac{Z^2}{1-Z}\right) + \ldots + \left(\frac{1}{1-Z} - \frac{Z^n}{1-Z}\right)\right]$$

$$= \frac{1}{n}\left[\frac{n}{1-Z} - \frac{Z(1+Z+\ldots+Z^{n-1})}{1-Z}\right.$$

$$\therefore \sigma_n = \frac{1}{1-Z} - \frac{Z(1-Z^n)}{n(1-Z)^2}$$

$$\lim_{n\to\infty} \sigma_n = \frac{1}{1-Z} - 0 = \frac{1}{1-Z} \quad (C,1)$$

Example 4.19:

Let $a_n = (-1)^{n+1}.\, n$

$S_n = a_1+a_2+\ldots = 1-2+3-4+\ldots$

$S_1=1;\ S_2 = -1;\ S_2 = 2;\ S_4 = -2\ldots$

$\{S_n\} = \{1,-1,2,-2,3,-3\ldots$

$\sigma_n = (S_1+S_2+\ldots\ldots+S_n)/n$

$\sigma_1 = 1;\ \sigma_2 = 0;\ \sigma_3 = 2/3;\ \sigma_4 = 0\ ;\ldots\ldots..$

(i.e.) $\sigma_n = \begin{cases} 1 & if\ n\ is\ even \\ \frac{\frac{(n+1)}{2}}{n} & if\ n\ is\ odd \end{cases}$

$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{n+1}{2n} = 1/2$ (if n is odd) & $\lim_{n\to\infty} \sigma_n=0$ (if n is even)

$\sum a_n$ is not (C,1) Summable

$a_n = \begin{cases} n & if\ n\ is\ odd \\ -n & if\ n\ is\ even \end{cases}$

'n' is even

$S_{2n} = a_1+a_2+\ldots+a_{2n}$

$= 1-2+3-4+\ldots\ldots\ldots. + (2n-1)-2n$

$= [1+3+5+\ldots\ldots\ldots+(2n-1)-2[1+2+\ldots\ldots. +n]$

$= n^2-2n(n+1)/2$

$S_{2n}= -n$

'n' is odd

$S_{2n+1} = a_1+a_2+\ldots\ldots+a_{2n+1}$

$= 1-2+3-\ldots\ldots-(2n) + (2n+1)$

$= -n+(2n+1)$

$= n+1$

$S_{2n+1} = n+1$

$\sigma_n=\frac{S_1+S_2+\ldots..+S_{2n}}{2n}$

$= \frac{1}{2n}(S_1+S_2+\ldots..+S_{2n-1})+(S_2+S_4+\ldots..+S_{2n})$

$= 1/2n\,(0) = 0$

$\sigma_{2n}=0$

$\lim_{n\to\infty} \sigma_n = 0$

(i.e.), $\lim_{n\to\infty} inf\ \sigma_n = 0$

$\sigma_{2n}=\frac{S_1+S_2+.....+S_{2n+1}}{2n+1}$

$= \frac{0+(n+1)}{2n+1}$

$\sigma_{2n+1}= \frac{n+1}{2n+1}$

$\lim_{n\to\infty} \sigma_{2n+1} = \lim_{n\to\infty} \frac{n(1+1/n)}{n(2+1/n)} = 1/2$

$\lim_{n\to\infty} \text{Sup}\ \sigma_n = 1/2$

Theorem 4.20:

If a series is convergent with sum S, then it is also (C,1) summable with Cesaro sum S.

Proof:

Let $\sum a_n$ be a convergent series with sum S.

Let S_n be the n^{th} partial sum of $\sum a_n$

let $\{\sigma_n\}$ be the sequence of arithmetic means defined by

$\sigma_n = \frac{S1+S_2+\cdots.+Sn}{n}$ (n=1,2,)……

To prove: $\{\sigma_n]$ converges & $\lim_{n\to\infty} \sigma_n = S$

Let $t_n = S_n\text{-}S$ & $\lim_{n\to\infty} \tau_n = \sigma_n\text{-}S$

Then

$\tau_n = \sigma_n\text{-}S = \frac{S1+S_2+\cdots.+Sn}{n}\ \text{-}S$

$= \frac{S1+S_2+\cdots.+Sn-nS}{n}$

$= \frac{(S1-S)+(S_2-S)+\cdots+(Sn-S)}{n}$

$\tau_n = \frac{t_1+t_2+\cdots+t_n}{n}$

To prove: $\lim_{n\to\infty} \tau_n = 0$

Given: $\sum a_n$ converges with sum S

The partial sum S_n converges to S

$\{t_n\}\to 0$ as $n\to\infty$

$\{t_n\}$ is a bounded sequence

Choose A> 0 so that $|t_n|\le A$ ……... (1)

Also $\{t_n\} \to 0$

given ε>0, choose N so that

$n>N \Rightarrow |t_n| < \varepsilon$ ………… (2)

For n>N

$|\tau_n| = |\frac{t_1+t_2+\cdots+t_n}{n}|$

$\le |\frac{t_1+t_2+\cdots+t_N}{n}| + |\frac{t_{N+1}+t_{N+2}+\cdots+t_n}{n}|$

$\le \frac{|t_1|+|t_2|+\cdots+|t_N|}{n} + \frac{|t_{N+1}|+|t_{N+2}|+\cdots+|t_n|}{n}|$

$< N.A/n + \varepsilon$ [by equation (1) & (2)]

(i.e.) $|\tau_n| < N.A/n + \varepsilon$

$\lim_{n\to\infty} \sup |\tau_n| \le \varepsilon$

E> 0 is arbitrary, we get

$\lim_{n\to\infty} |\tau_n| = \varepsilon$

(i.e.) $\lim_{n\to\infty} \tau_n = 0$

(i.e.), $\lim_{n\to\infty} \sigma_n - S = 0$

$\Rightarrow \lim_{n\to\infty} \sigma_n = S$

$\sum a_n$ is cesaro summable with cesaro sum S.

Note:

If a sequence $\{S_n\}$ converges, then the sequence $\{\sigma_n\}$ of arithmetic means also converges to the same limit, (i.e.),$\{S_n\} \rightarrow S = \{\sigma_n\} \rightarrow S$

Note:

Cesaro summability is one of the large class of Summability methods which can be used to assign a 'sum' to an 'infinite series'

Infinite Products:

Definition 4.21:

Given a sequence $\{u_n\}$ of real or complex numbers,

Let $p_1 = u_1$, $P_2 = , u_1u_2$, $P_n = u_1u_2\ldots.u_n \prod_{k=1}^{n} u_k$

The ordered pair of sequences $(\{u_n\}, \{P_n\})$ is called an infinite product (or simply, a product).

The number P_n is called the n^{th} partial product and u_n is called the n^{th} factor of the product. The following symbols are used to denote the product

$u_1u_2\ldots.u_n \prod_{k=1}^{n} u_k$

Note:

The symbol $\prod_{n=N+1}^{\infty} u_n$ means $\prod_{n=1}^{\infty} u_{N+n}$ .We can write $\prod u_n$ .If $\{P_n\}$ converges, then the infinite product, $\prod_{n=1}^{\infty} u_n$ converges

Definition 4.22:

Given an infinite product $\prod_{n=1}^{\infty} u_{n,}$ let $P_n = \prod_{k=1}^{n} u_k$

(a) If infinitely many factors u_n are zero, we say the product diverges to zero.

(b) If no factor u_n is zero, we say the product converges if there exists a number $p \neq 0$ such that $\{P_n\}$ converges u_n to P. In this case, p is called the value of the product and we write $p = \prod_{n=1}^{\infty} u_n$.

If $\{P_n\} \rightarrow o$, we say that the product diverges to '0'

(c) If there exists an N such that $n > N \Rightarrow u_n \neq 0$, we say $\prod_{n=1}^{\infty} u_n$ converges, provided that $\prod_{n=1}^{\infty} u_n$ converges as described in (b). In this case, the value of the product $\prod_{n=1}^{\infty} u_n$ is $u_1, u_2, \ldots u_N \prod_{n=N+1}^{\infty} u_n$

(d) $\prod_{n=1}^{\infty} u_n$ is called divergent if it does not converge as described in (b) or (c)

Note:

- The value of a convergent Infinite product is zero $\Leftrightarrow$ A finite number of factors are zero.
- The convergence of an infinite product is not affected by inserting or removing a finite number of factors, zero or not.
- $\prod_{n=1}^{\infty} a_n$ converges when the limit exists & is not zero. Otherwise $\prod_{n=1}^{\infty} a_n$

Example 4.23:

$\prod_{n=1}^{\infty}(1+\frac{1}{n})$ & $\prod_{n=1}^{\infty}(1-\frac{1}{n})$ are both divergent.

(i) $P_n = (1+\frac{1}{1}).(1+\frac{1}{2}) \ldots\ldots (1+\frac{1}{n}) = \frac{2}{1}, \frac{3}{2} \ldots\ldots \frac{n}{n-1}. n+\frac{1}{n} = n+1$

(ii) $P_n = \frac{1}{2}. \frac{2}{3} \ldots\ldots \frac{n-2}{n-1}. \; n\text{-}\frac{1}{n} = \frac{1}{n}$

In this first case $P_n = n+1$ & In the second case $P_n = \frac{1}{n}$

Theorem 4.24: [Cauchy condition for Products]

The Infinite product $\prod u_n$ converges if and if for every $\varepsilon > 0$, there exist N such that $n > N \Rightarrow |u_{n+1}.u_{n+2} \ldots\ldots u_{n+k} - 1| < \varepsilon$ for $K = 1,2,3,\ldots$

Proof:

Assume that $\prod u_n$ converges

Assume that no u_n is zero

Let $P_n = u_1.u_2 \ldots\ldots u_n$ & $P = \lim_{n\to\infty} P_n$

Since, $u_n \neq 0 \Rightarrow P_n \neq 0 \Rightarrow P \neq 0$

$\therefore$ there exist $M > 0$ such that $|P_n| > M$ …. (1)

Now. $\{P_n\}$converges

$\{P_n\}$ satisfies the Cauchy condition for sequence

given ε>0, there exist N such that

$n>N \Rightarrow |P_{n+k} - P_n| < N\ M.\ \varepsilon \quad k=1,2,3\ldots\ldots \qquad \ldots\ldots.(2)$

Now,

$\left|\frac{P_{n+K}-P_n}{P_n}\right| = \left|\frac{P_{n+k}}{P_n} - \right|$

$=\left|\frac{u_1 u_2 \ldots\ldots u_n u_{n+1} \ldots\ldots u_{n+k}}{u_1 u_2 \ldots\ldots u_n} - 1\right|$

$\left|\frac{P_{n+K}-P_n}{P_n}\right| = |u_{n+1}u_{n+2}\ldots\ldots u_{n+k} - 1| \quad \ldots\ldots(*)$

$|u_{n+1}u_{n+2}\ldots\ldots u_{n+k} - 1| = \left|\frac{P_{n+K}-P_n}{P_n}\right|$

$= \frac{|P_{n+K}-P_n|}{|P_n|}$

$< M.\ \varepsilon/M$ (by equation (1) & (2))

$|u_{n+1}u_{n+2}\ldots\ldots u_{n+k} - 1| < \varepsilon$ for k= 1,2, ……

Conversely,

Assume that for all ε>0 there exist N such that

$n>N \Rightarrow |u_{n+1}u_{n+2}\ldots\ldots u_{n+k} - 1| < \varepsilon$

To prove: $\prod u_n$ converges

Then $n>N \Rightarrow u_n \neq 0$

Sup $u_n=0$

From equation (2) $\Rightarrow |u_{n+1}u_{n+2}\ldots\ldots u_{n+k} - 1| < \varepsilon$

$\Rightarrow |0-1| < \varepsilon$

$\Rightarrow \varepsilon > 1$ which is Impossible

$\therefore u_n \neq 0$

Take ε = ½ in 2,

we get $|u_{n+1}u_{n+2} \ldots\ldots u_{n+k} -1| < \frac{1}{2}$

Let N_0 be the corresponding value of N &

let $a_n = u_{N0+1}u_{N0+2} \ldots\ldots u_{n+k}$ If $n>N_0$

From equation (3) $\Rightarrow |u_{N0+1}u_{N0+2} \ldots\ldots u_{n+k} -1| < \frac{1}{2}$

$\Rightarrow |q_n-1| < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < q_n-1 < \frac{1}{2}$

$\Rightarrow -\frac{1}{2}+1 < q_n < \frac{1}{2}+1$

$\Rightarrow \frac{1}{2} < |q_n| < \frac{3}{2}$ ……... (4)

If $\{a_n\}$ converges, it cannot converge to zero

To show that $\{q_n\}$ converges

Let ε > o be given

(*) $\Rightarrow \left|\frac{q_{n+k}-q_n}{q_n}\right| < \varepsilon$

$\Rightarrow \frac{|q_{n+k}-q_n|}{|q_n|} < \varepsilon$

$\Rightarrow |q_{n+k} -q_n| < \varepsilon\, |q_n| < \varepsilon.3/2$ (by equation (4))

$\Rightarrow |q_{n+k} - q_n| < 3\varepsilon/2$

$\{q_n\}$ satisfies the Cauchy condition for sequences

$\therefore \{q_n\}$ converges

$\therefore \prod u_n$ converges

Note:

Take K=1 in Cauchy condition for Product, we get $\prod u_n$ converges = $\lim_{n\to\infty} a_n = 0$

∴ We can write $u_n = 1+a_n$

Thus $\prod(1+a_n)$ converges $\lim_{n\to\infty} a_n = 0$

Theorem 4.25:

Assume that each $a_n>0$. Then the product $\prod$ $(1+a_n)$ converges if and only if the series $\sum a_n$ converges

Proof:

Assume that $a_n>0$

Let $1+x \leq e^x \ \forall\ x$ ..……. (1)

Let $x \geq 0$

When x>0, by Mean-Value Theorem, we get,

$[f(x)-f(0) = f'(x_0)(x-0)]$

(i.e.)$e^x-e^0 = (e^{x_0})'(x-0), \quad 0<x_0<x$

$\Rightarrow e^x-1 = e^{x_0}.x$

$\Rightarrow e^x-1= x.e^{x_0}$ where $0<x_0<x$ ………(2)

We know that

$e^{x_0}\geq 1$

From equation (2) $\Rightarrow e^x-1 = xe^{x_0}$

$\geq x.1$

$e^x-1 \quad \geq x$

Let $S_n=a_1+a_=+……+a_n$

$P_n =(1+a_1)(1+a_2) …... (1+a_n)$

Clearly, $\{S_n\}$ & $\{P_n\}$ are both increasing

To show that $\{S_n\}$ bounded above $\Rightarrow\{P_n\}$ bounded above

Clearly, $a_1 + a_2 + \ldots + a_n < a_1 . a_2 \ldots a_n$

$< (1+a_1)(1+a_2) \ldots (1+a_n)$

(i.e.) $S_n < P_n$ ……. (3)

Take $x = a_k$ to 1, where k = 1,2, ……n

$1 \Rightarrow 1 + a_k \leq e^{a_k}$ k=1,2,…n

$(1+a_1)(1+a_2)\ldots(1+a_n) \leq e^{a_1} . e^{a_2} \ldots\ldots\ldots\ldots\ldots e^{a_n}$

$(1+a_1)(1+a_2)\ldots(1+a_n) \leq e^{a_1 + a_2 \ldots + a_n}$

$\Rightarrow p^n < e^{S_n}$ ………. (4)

From equation (3) & (4) we get,

$\{S_n\}$ is bounded $\Leftrightarrow \{P_n\}$ is bounded above

$\{S_n\}$ converges $\Leftrightarrow$ $\{P_n\}$ converges.

$\sum a_n$ converges $\Leftrightarrow \prod (1+a_n)$ converges [$\because \{S_n\}$ & $\{P_n\}$ both increasing]

(i.e.), $\prod (1+a_n)$ converges $\Leftrightarrow \sum a_n$ converges

Note:

In the above Theorem $\{P_n\}$ cannot converges to zero

Since each $P_n \geq 1$

Also $P_n \to \infty$ if $S_n \to \infty$

Definition 4.26:

The product $\prod (1+an)$ is said to converge absolutely if $\prod (1+|a_n|)$ converges

Theorem 4.27:

Absolute convergence of $\prod (1+a_n)$ implies convergence

Proof:

Assume that $\prod (1+|a_n|)$ converges

To prove: $\prod (1+a_n)$ converges.

Now, $\prod (1+|a_n|)$ converges

By Cauchy Condition for Product Theorem 4.24 we get, $\forall\ \varepsilon >0$, there exist N such that n>N $\Rightarrow |(1+|a_{n+1}|)(1+|a_{n+2}|)\ldots\ldots(1+|a_{n+k}|)-1| < \varepsilon$ for k = 1,2,….

Now,

$|(1+a_{n+1})\ (1+a_{n+2})\ \ldots\ (1+a_{n+k}) - 1\ |$

$\leq (1+|a_{n+1}|)(1+|a_{n+2}|)\ldots\ldots(1+|a_{n+k}|)-1$

$\leq |(1+|a_{n+1}|)(1+|a_{n+2}|)\ldots\ldots(1+|a_{n+k}|-1|$

$<\varepsilon$ (by equation (1))

$\therefore\ |(1+|a_{n+1}|)\ (1+|a_{n+2}|)\ \ldots\ldots\ (1+|a_{n+k}|-1| < \varepsilon$ for k=1,2,…..

$\therefore \prod (1+a_n)$ converges

Note:

- $\prod (1+|a_n|)$ converges if and only if $\sum |a_n|$ converges

(i.e.) $\prod (1+a_n)$ converges absolutely if and only if $\sum a_n$ converges absolute

Theorem 4.28:

Assume that each $a_n>0$. Then the product $\prod (1-a_n)$ converges if and only if the series $\sum a_n$ converges.

Proof:

Assume that $a_n>0$

Suppose $\sum a_n$ converges

$\prod$ **(1**-a_n) converges absolutely

$\Rightarrow \prod (1-a_n)$ Converges

Conversely,

Assume $\prod (1-a_n)$ converges

To prove: $\sum a_n$ converges

Suppose that $\sum a_n$ diverges,

If $\{a_n]$ does not converge to zero, the $\prod (1-a_n)$ also diverges

We may assume that $a_n \to 0$ as $n\to\infty$

By discarding finitely many terms,

we may assume that $a_n \leq 1/2$ for all $n\geq 1$

$1-a_n \geq 1/2 \ \forall n\geq 1$

$\therefore 1-a_n \neq 0 \quad \forall\ n\geq 1$

Let $p_n = (1-a_1)(1-a_2) \ldots\ldots. (1-a_n)$ and

$q_n = (1+a_1)(1+a_2) \ldots. (1+a_n) \quad \forall\ n\geq 1$

Then

$(1-a_k)(1+a_k) = 1- a_k^2 \leq 1 \quad (k=1,2,\ldots n)$

$(1-a_k)(1+a_k) \leq 1$

$\Rightarrow p_n q_n \leq 1 \ \forall\ n\geq 1$

$\Rightarrow p_n \leq 1/q_n \ \forall\ n\geq 1$

$\therefore \sum a_n$ diverges, then $\prod (1+a_n)$ diverges (by Theorem8.52)

$\therefore q_n \to\infty$ an $n\to\infty$

$\therefore 1/q_n \to 0$ as $n\to\infty$

(i.e.) $p_n \to 0$ as $n\to\infty$

$\therefore \prod_{n=1}^{\infty} (1-a_n)$diverges to 0

$\therefore$our assumption is wrong

$\therefore \sum a_n$ converges

Power Series

Definition 4.29:

An infinite series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ ......(1) is called a power series in $z-z_0$. Here z, z_0 and $a_n(n=0,1,...)$ are complex numbers.

- With every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, there is associated a disk, called the disk of convergence, such that the series converges absolutely for every z interior to this disk and diverge for every z outside this disk.
- The centre of the disk is at z_0.
- The radius of the disk is called the radius of convergence of the power series. (The radius may be o or +∞ in extreme cases).

Theorem 4.30:

Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, let $\lambda = \lim_{n\to\infty} sup \sqrt[n]{|a_n|}$, $r=\frac{1}{\lambda}$, (where $r = 0$ if $\lambda = +\infty$ and $r = +\infty$ if $\lambda = 0$). Then the series converges absolutely if $|z-z_0| < r$ and diverges if $|z-z_0| > r$. Furthermore, the series converges uniformly on every compact subset interior to the disk of convergence.

Proof:

Given $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series.

Let $\lambda = \lim_{n\to\infty} sup \sqrt[n]{|a_n|}$

Let $r = \frac{1}{\lambda}$

First to prove that $\sum a_n(z-z_0)^n$ converges absolutely if $|z-z_0| < r$ and $\sum a_n(z-z_0)^n$ diverges if $|z-z_0| > r$.

Now, $r = \frac{1}{\lambda}$

$$\lambda = \frac{1}{r}$$

$$\lim_{n\to\infty} sup \sqrt[n]{|a_n|} = \frac{1}{r}$$

$$\lim_{n\to\infty} sup \sqrt[n]{|a_n(z-z_0)^n|} = \frac{|z-z_0|}{r}$$

By Root test theorem,

$\sum a_n(z-z_0)^n$ converges absolutely if $|z-z_0| < r$ and $\sum a_n(z-z_0)^n$ diverges if $|z-z_0| > r$.

Let T be a compact subset of the disk of convergence there exists $p \in T$ such that $z \in T$ implies

$|z-z_0| \le |p-z_0| < r$

$|a_n(z-z_0)^n| \le |a_n(p-z_0)^n| \; for \; all \; z \in T$

We have $|a_n(p-z_0)^n|$ converges.

Therefore, by Weiestrass M – test, we get

$\sum a_n(z-z_0)^n$ converges uniformly.

Note:

If the limit $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists (or if this limit is $+\infty$) its value is also equal to the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Example 4.31:

The two series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty}(z^n/n^2)$ have the same radius of convergence, namely r = 1.

On the boundary of the disk of convergence $|z-z_0| = r$, $\sum_{n=0}^{\infty} z^n$ converges nowhere and $\sum_{n=0}^{\infty}\frac{z^n}{n^2}$ converges everywhere.

For, $\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} 1(z-0)^n$

Here $a_n = 1$

Therefore, Radius of convergence of $\sum_{n=0}^{\infty} z^n = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1$

Also, $\sum_{n=0}^{\infty}\frac{z^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}(z-z_0)^n$

Here $a_n = \frac{1}{n^2}$

Therefore, radius of convergence of

$$\sum_{n=0}^{\infty}\frac{z^n}{n^2} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1/n^2}{1/(n+1)^2}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^2\right| = 1$$

Example 4.32:

Consider the series $\sum_{n=0}^{\infty}\frac{z^n}{n}$

$$\sum_{n=0}^{\infty}\frac{z^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}(z-0)^n$$

Here $a_n = \frac{1}{n}$

∴ Radius of convergence

$$= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \lim_{n\to\infty}\left|1+\frac{1}{n}\right| = 1$$

$\therefore r = 1$

By Dirichlet Test theorem , $\sum_{n=0}^{\infty}\frac{z^n}{n}$ converge everywhere else on the boundary.

Theorem 4.33:

Assume that the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for each $z \in B(z_0; r)$. Then the function 'f' defined by the equation $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, if $z \in B(z_0; r)$ is continuous on $B(z_0; r)$.

Proof:

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} f_n(z)$ for all $z \in B(z_0; r)$ where $f_n(z) = a_n(z-z_0)^n$.

Given that $f(z)$ converges for each $z \in B(z_0; r)$

Each point in $B(z_0; r)$ belongs to some compact subset of $B(z_0; r)$.

Let that compact subset be 's'.

$\therefore f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} f_n(z)$ converges uniformly on 's' …….(1) Now, $f_n(z) = a_n(z-z_0)^n$ is a polynomial function

Since every polynomial function is continuous,

$'f_n'$ is continuous on S ……… (2)

By equation (1) and (2) also using theorem 5.14, we get

'f' is continuous on every compact subset 'S' of $B(z_0; r)$

Hence 'f' is continuous on $B(z_0; r)$.

Theorem 4.34:

Assume that $\sum a_n(Z - Z_o)^n$ converges if $Z \in B(Z_o; r)$. Suppose that the equation $f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_o)^n$, is known to be valid for each $'Z'$ in some open subset $'S'$ of $B(Z_o; r)$ .Then for each point Z_1 in S, there exists a neighbourhood $B(Z_1; r) \subseteq S$ in which 'f' has a power series expansion of the form $f(Z) = \sum_{k=0}^{\infty} b_k(Z - Z_1)^k$,

where $b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (Z_1 - Z_o)^{n-k} \quad (k = 0,1,2, \dots)$

Proof:

Assume that $\sum a_n(Z - Z_o)^n$ converges if $Z \in B(Z_o; r)$ ……… (1)

Given: $f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_o)^n$ is valid $\forall\, Z \in S \subseteq B(Z_o; r)$

To Prove: $\forall\, Z_1 \in S$, there exists $B(Z_1; r) \subseteq S$

such that $f(Z) = \sum_{k=0}^{\infty} b_k(Z - Z_1)^k$,

where $b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (Z_1 - Z_o)^{n-k} \quad (k = 0,1,2, \dots)$

Now, $Z \in S$ we have

$$f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_o)^n$$

$f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_1 + Z_1 - Z_o)^n$ ……… (2)

$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n}{k} (Z - Z_1)^k (Z_1 - Z_o)^{n-k}$

$$\left[\because (a + b)^n = \sum_{r=0}^{n} n_{C_r}\, a^{n-r} b^r\right]$$

$f(Z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_n(k)$ ………… (3)

Where $C_n(k) = \begin{cases} \binom{n}{k} a_n (Z - Z_1)^k (Z_1 - Z_o)^{n-k} & if\ k \le n \\ 0 & if\ k > n \end{cases}$

Choose R so that $B(Z_1; R) \subseteq S$ and assume that $Z \in B(Z_1; R)$

Claim: $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_n(k)$ converges absolutely

Now, $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C_n(k)| = |f(Z)|$

$$= |\sum_{n=0}^{\infty} a_n(Z - Z_1 + Z_1 - Z_o)^n| \quad \text{(by(1))}$$

$$= \sum_{n=0}^{\infty} |a_n|(|Z - Z_1| + |Z_1 - Z_o|)^n$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C_n(k)| = \sum_{n=0}^{\infty}|a_n|(Z_2 - Z_o)^n \quad \ldots\ldots\ldots (4)$$

Where $Z_2 = Z_o + |Z - Z_1| + |Z_1 - Z_o|$

Now, $|Z_2 - Z_o| = |Z - Z_1| + |Z_1 - Z_o|$

$$< R + |Z_1 - Z_o| \qquad (\because Z \in B(Z_1; R))$$

$$\leq r$$

$\therefore |Z_2 - Z_o| \leq r$

$i.e; Z_2 \in B(Z_o; r)$

$\therefore \sum_{n=0}^{\infty}|a_n|(Z_2 - Z_o)$ Converges (by $equation$ (1))

(i.e) $equation$ (4) becomes $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C_n(k)|$ converges

(i.e) $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_n(k)$ converges absolutely

$\therefore$ By theorem 4.10, We get $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_n(k) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C_n(k)$

$\therefore$ (3) $\Rightarrow f(Z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_n(k)$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C_n(k)$$

$$= \sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}a_n(Z - Z_1)^k(Z_1 - Z_o)^{n-k}$$

$\therefore f(Z) = \sum_{k=0}^{\infty} b_k(Z - Z_1)^k$,

where $b_k = \sum_{n=k}^{\infty}\binom{n}{k}a_n(Z_1 - Z_o)^{n-k} \quad (k = 0,1,2,\ldots)$

Note:

In the course of the proof, we have shown that we may use any $R > 0$ that satisfies the condition $B(Z_1; r) \subseteq S$

Theorem 4.35:

Assume that $\sum a_n(Z - Z_o)^n$ converges for each Z in $B(Z_o; r)$. Then the function 'f' defined by the equation $f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_o)^n$, if $Z \in B(Z_o; r)$ has a derivative $f'(Z)$ for each Z in $B(Z_o; r)$, given by $f'(Z) = \sum_{n=0}^{\infty} n\, a_n(Z - Z_o)^{n-1}$

Proof:

Assume that $\sum a_n(Z - Z_o)^n$ converges for each $Z \in B(Z_o; r)$

Define 'f' by $f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_o)^n$, if $Z \in B(Z_o; r)$ .... (1)

To Prove: $f(Z)$ has a derivative $f'(Z)$ such that $f'(Z) = \sum_{n=0}^{\infty} n\, a_n(Z - Z_o)^{n-1}$ $\forall Z \in B(Z_o; r)$

Assume that $Z_1 \in B(Z_o; r)$ if $Z \in B(Z_1; R)$, $Z \neq Z_1$, We have

$f(Z) = \sum_{k=0}^{\infty} b_k (Z - Z_1)^k$, where $b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (Z_1 - Z_o)^{n-k}$ ……… (2)

Now,

$$f(Z) = \sum_{k=0}^{\infty} b_k (Z - Z_1)^k$$

$$\Rightarrow f(Z) = b_o (Z - Z_1)^0 + b_1 (Z - Z_1)^1 + \sum_{k=2}^{\infty} b_k (Z - Z_1)^k$$

$$\therefore f(Z) = b_o + b_1 (Z - Z_1) + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^{k+1} \quad \text{……….. (3)}$$

Now,

$b_o = \sum_{n=0}^{\infty} \binom{n}{0} a_n (Z_1 - Z_o)^{n-0}$ (by equation (2))

$\Rightarrow b_o = \sum_{n=0}^{\infty} 1. a_n (Z_1 - Z_o)^n$

$i.e; b_o = \sum_{n=0}^{\infty} a_n (Z_1 - Z_o)^n$

$\Rightarrow b_o = f(Z_1)$ (by equation (1))

$$\therefore (3) \Rightarrow f(Z) = f(Z_1) + b_1 (Z - Z_1) + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^{k+1}$$

$$\Rightarrow f(Z) - f(Z_1) = b_1 (Z - Z_1) + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^{k+1}$$

$$\Rightarrow f(Z) - f(Z_1) = (Z - Z_1)[b_1 + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^k]$$

$$\Rightarrow \frac{f(Z) - f(Z_1)}{(Z - Z_1)} = b_1 + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^k$$

$$\Rightarrow \lim_{Z \to Z_1} \frac{f(Z) - f(Z_1)}{(Z - Z_1)} = \lim_{Z \to Z_1} [b_1 + \sum_{k=1}^{\infty} b_{k+1} (Z - Z_1)^k]$$

$$= b_1$$

$i.e; f'(Z_1) = \lim_{n \to \infty} \frac{f(Z) - f(Z_1)}{(Z - Z_1)} = b_1$

Hence $f'(Z_1)$ exists and $f'(Z_1) = b_1$ ……….. (4)

Now, (2) becomes $b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n (Z_1 - Z_o)^{n-k}$

$$\Rightarrow b_1 = \sum_{n=1}^{\infty} \binom{n}{1} a_n (Z_1 - Z_o)^{n-1}$$

$$\Rightarrow b_1 = \sum_{n=1}^{\infty} n a_n (Z_1 - Z_o)^{n-1}$$

$(4) \Rightarrow f'(Z_1) = \sum_{n=1}^{\infty} n a_n (Z_1 - Z_o)^{n-1}$

$\therefore$ 'Z_1' is an arbitrary point of $B(Z_o; r)$, then $f'(Z) = \sum_{n=1}^{\infty} n a_n (Z - Z_o)^{n-1}$

Note: [Hadamard's Formula]

The radius of convergence of the power series $\sum a_n (Z - Z_o)^n$ is given by

R.O.C= $\left(\lim_{n \to \infty} sup \sqrt[n]{|a_n|}\right)^{-1}$

Theorem 4.36:

The power series $f(Z) = \sum_{n=0}^{\infty} a_n (Z - Z_o)^n$ and the derivative of a power series $f'(Z) = \sum_{n=1}^{\infty} na_n (Z - Z_o)^{n-1}$ have the same radius of convergence.

Proof:

Given: $f(Z) = \sum_{n=0}^{\infty} a_n (Z - Z_o)^n$ and $f'(Z) = \sum_{n=1}^{\infty} na_n (Z - Z_o)^{n-1}$

Now, $f'(Z) = \sum_{n=1}^{\infty} na_n (Z - Z_o)^{n-1}$

$= \sum_{n=0}^{\infty} na_n (Z - Z_o)^{n-1}$

$= \frac{1}{Z-Z_o} \sum_{n=0}^{\infty} na_n (Z - Z_o)^n$

$\therefore \; f'(Z) = \frac{1}{Z-Z_o} \sum_{n=0}^{\infty} na_n (Z - Z_o)^n$ .... (2)

Clearly, both the series (1) and (2) converge for the same values of 'Z'

$\therefore$ We apply Hadamard formula in (2)

R.O.C of $f'(Z) = \left(\lim_{n\to\infty} sup \sqrt[n]{|na_n|}\right)^{-1}$

$= \left(\lim_{n\to\infty} sup \sqrt[n]{|a_n|}\right)^{-1}$ $\left[\because \lim_{n\to\infty} \sqrt[n]{n} = 1\right]$

$=$ R.O.C of $f(Z)$

$\therefore$ R.O.C of $f'(Z) =$ R.O.C of $f(Z)$

Note:

By repeated application of $f'(Z) = \sum_{n=1}^{\infty} na_n (Z - Z_o)^{n-1}$, we find that for each $k = 0,1,2, \dots$, the derivative $f^k(Z)$ exists in $B(Z_o; r)$ and is given by the series

$f^{(k)}(Z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (Z - Z_o)^{n-k}$ .... (1)

Put $Z = Z_o$ in (1), we get

$f^{(k)}(Z_o) = k!\, a_k \qquad k = 0,1,2, \dots$ .... (2)

This equation tells us that if two power series $\sum a_n (Z - Z_o)^n$ and $\sum b_n (Z - Z_o)^n$ both represent the same function in a neighbourhood $B(Z_o; r) \therefore a_n = b_n \quad \forall\, n$

$(i.e.)$ The power series expansion of a function 'f' about a given point Z_o is uniquely determined and $f(Z) = \sum_{n=0}^{\infty} a_n (Z - Z_o)^n$ becomes $f(Z) = \sum_{n=0}^{\infty} \frac{f^n(Z_o)}{n!} (Z - Z_o)^n$ (by(2)) valid for each Z in the disk of convergence.

Multiplication of Power Series

Theorem 4.37:

Given two power series expansions about the origin, say

f (z) = $\sum_{n=0}^{\infty} a_n z^n$, if $z \epsilon B(0;r)$ & g (z) = $\sum_{n=0}^{\infty} b_n z^n$, if $z \epsilon B(0;R)$

Then the product $f(z), g(z)$ is given by the power series

$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, if\ z \epsilon B(0;r) \cap B(0;R)$ where $c_n = \sum_{n=0}^{\infty} a_k b_{n-k}$

$(n = 0,1,2, \dots)$

Proof:

Given $f\ (z)$ = $\sum_{n=0}^{\infty} a_n z^n$, if $z \epsilon B(0;r)$ …… (1)

$g\ (z)$ = $\sum_{n=0}^{\infty} b_n z^n$, if $z \epsilon B(0;R)$ …… (2)

Then $f\ (z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely (with sum f(z))

& $g(z) = \sum_{n=0}^{\infty} b_n z^n$ converges absolutely (with sum g(z))

The Cauchy product of two series (1) & (2) is

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k z^k b_{n-k} z^{n-k}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

$$= \sum_{n=0}^{\infty} c_n z^n$$

where $c_n = \sum_{n=0}^{\infty} a_k b_{n-k}$

Here $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely with sum f(z) on $B(0;r)$ and $\sum_{n=0}^{\infty} b_n z^n$ converges absolutely with sum g(z) on B(0;R).

Then by Merten's theorem 4.13,

The Cauchy product $\sum_{n=0}^{\infty} c_n z^n$ converges with sum $f(z)g(z)$ on $B(0;r) \cap B(0;R)$.

(i.e.) $\sum_{n=0}^{\infty} c_n z^n = f(z)g(z)$ if $z \epsilon B(0;r) \cap B(0;R)$.

Hence $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$ if $z \epsilon B(0;r) \cap B(0;R)$

where $c_n = \sum_{n=0}^{\infty} a_k b_{n-k}$ $(n = 0,1,2, \dots)$

The Taylor's series generated by a function

Definition 4.38:

Let f be a real-valued function defined on an interval I in R. If f has derivatives of every order at each point of I, we write $f\epsilon c^{\infty}$ on I

Note:

- If $f\epsilon c^{\infty}$ on some neighborhood of a point c, then the power series

  $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n}$ ……. (1)

  is called the Taylor's series about c generated by f.
- To indicate that f generates this series, we write

  $f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n}$ … …. (2)
- Taylor's Formula states that if $f\epsilon c\infty$ on the closed interval [a,b], then for every xϵ[a, b] & for every n, we have

  f(x) = $\sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^{k} + \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n}$ …….. (3) , there exists $x_1\epsilon[x,c]$.
- The point x_1 depends on x, c & on n.
- Hence a necessary and sufficient condition for the Taylor's series to converge to f(x) is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n} = 0$ ……… (4)
- In practice it may be quite difficult to deal with this limit because of the unknown position of x_1.
- In some cases, however, a suitable upper bound can be obtained for $f^{(n)}(x_1)$ and the limit can be shown to be zero.
- Since $\frac{A^{n}}{n!}\to 0$ as n$\to\infty$ $\forall$ n, (4) will hold if there exists

  $M > 0 \ni: |f^{(n)}(x)| \leq M^{n} \ \forall\ x\epsilon[a,b]$.
- In other words, the Taylor's series of a function f converges if the nth derivative $f^{(n)}$ grows go faster than the nth power of some positive integer.

Theorem 4.39

Assume that $f\epsilon c^{\infty}$ on [a, b] and let $c\epsilon[a,b]$. Assume that the is a neighborhood B(C) and a constant M (which might depend on C) such that $|f^{(n)}(x)| \leq M^{n}$ for every $each\ x\ \epsilon B(C) \cap$

$[a,b]$ and every $n = 1,2, \ldots..$ Then, for each $x \epsilon B(C) \cap [a,b]$, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Proof:

Given, $f \epsilon c^{\infty}$ on [a, b] and $c \epsilon [a,b]$

Then by Taylors formula,

$\forall\, x \epsilon [a,b]$ and $\forall n$, we have

$f(x) = \sum_{n=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(x_1)}{n!}(x-c)^n$, there exists $x_1 \in [x_2, c]$

…….. (1)

Assume that there exists a neighbourhood $B(C)$ and a constant M (depends on C) $\ni : |f^{(n)}(x)| \leq M^n \;\forall\, x \epsilon [a,b]$

Clearly, $x_1 \epsilon B(C) \cap [a,b]$ therefore, $|f^{(n)}(x_1)| \leq M^n$

$\Rightarrow\; -M^n \leq f^{(n)}(x_1) \leq M^n$

$\Rightarrow\; f^{(n)}(x_1) \leq M^n$

$\Rightarrow \frac{f^{(n)}(x_1)}{n!} \leq \frac{M^n}{n!}$

We know that, $\frac{M^n}{n!} \to 0 \; as \; n \to \infty \;\forall\, M$

Therefore, by comparison test

$\frac{f^{(n)}(x_1)}{n!} \to 0 \; as \; n \to \infty$

$\therefore \frac{f^{(n)}(x_1)}{n!}(x-C)^n \to 0 \; as \; n \to \infty$

That is, $\frac{f^{(n)}(x_1)}{n!}(x-C)^n = 0$ ………(2)

Now (1) $\Rightarrow \lim_{n\to\infty} f(x) = \lim_{n\to\infty} \sum_{n=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \lim_{n\to\infty} \frac{f^{(n)}(x_1)}{n!}(x-c)^n$

$\Rightarrow f(x) = \lim_{n\to\infty} \sum_{n=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + 0$ by (2)

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n}$

(i.e.) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n} \quad \forall\, x \in B(C) \cap [a,b]$.

Berstein's Theorem

Theorem 4.40:

Assume f has a continuous derivative of order n+1 in some open interval I containing c, and define $E_n(x)$ for x in I by the equation

$f(x) = \sum_{k=0}^{n} \frac{f^{(k))}(c)}{k!}(x-c)^{k} + E_n(x)$. Then $E_n = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)\, dt$

Proof:

Assume f has a continuous derivative of order n+1 in some open interval I containing 'c'

Define: E_n for x in I by $f(x) = \sum_{k=0}^{n} \frac{f^{(k))}(c)}{k!}(x-c)^{k} + E_n(x)$ ………..(1)

To prove: $E_n = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)\, dt$

We prove the theorem by induction on n

For n=1

Equation (1) $\Rightarrow f(x) = \sum_{k=0}^{n} \frac{f^{(k))}(c)}{k!}(x-c)^{k} + E_n(x)$

$f(x) = \frac{f^{(0)}(c)}{0!}(x-c)^0 + \frac{f^{(1)}(c)}{1!}(x-c)^1 + E_1(x)$

$f(x) = f(c) + f'(c)(x-c) + E_1(x)$

$E_1(x) = f(x) - f(c) - f'(c)(x-c)$

$E_1(x) = \int_c^x [f'(t) - f'(c)]\,dt$

$E_1(x) = \int_c^x u(t)dv(t)$ where $u(t) = f'(t) - f'(c)$; $v(t) = t - x$

$E_1(x) = u(t)v(t)]_c^x - \int_c^x v(t)du(t)$ $\quad [\int udv = uv - \int vdu]$

$E_1(x)=u(x)v(x)-u(c)v(c)-\int_c^x (t-x)f''(t)dt$

$E_1(x)=[0-0]-\int_c^x (t-x)\, f''(t)$ dt

$E_1(x)=\int_c^x (x-t)\, f''(t)$ dt

The result is true for n=1

Now, we assume that the result is true for 'n'

(i.e.) $E_n = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)$ dt ....(2)

Now, to prove the result is true for 'n+1'

From equation (1) ⇒ $f(x) = \sum_{k=0}^{n+1} \frac{f^{(k))}(c)}{k!}(x-c)^k + E_{n+1}(x)$

$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + E_{n+1}(x)$

$E_{n+1}(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$

$E_{n+1}(x) = E_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$ (by equation (1))

$E_{n+1} = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)$ dt $- \frac{f^{n+1}(c)}{(n+1)!}\int_c^x (n+1)(x-t)^n$ dt

$E_{n+1} = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)$ dt $- \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(c)$ dt

$E_{n+1} = \frac{1}{n!}\int_c^x (x-t)^n [f^{(n+1)}(t)-f^{(n+1)}(c)]$ dt

$E_{n+1}(x)= \int_c^x u(t)dv(t)$ where $u(t) = f^{(n+1)}(t)-f^{(n+1)}(c)$; $dv(t) = (x-t)^n$. dt

$E_{n+1}(x)=u(t)v(t)]_c^x - \int_c^x v(t)du(t)$ $[\int udv = uv - \int vdu]$

$E_{n+1}(x)=\frac{1}{n!}[u(x)v(x)-u(c)v(c)] - \int_c^x \frac{-(x-t)^{\,n+1}}{n+1} f^{(n+2)}(t)dt$

$[v(t) = \frac{-(x-t)^{\,n+1}}{n+1}$; $du(t) = f^{(n+2}(t)dt]$

$E_{n+1}(x)=\frac{1}{n!}[0-0] - \int_c^x \frac{-(x-t)^{\,n+1}}{n+1} f^{(n+2)}(t)dt$

$E_{n+1}(x) = \frac{1}{(n+1)!}\int_c^x (x-t)^{n+1} f^{(n+2)}(t)dt$

Therefore, the result is true for n+1

Hence $E_n = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)\, dt$

Note:

The change of variable t= x+(c-x) u transforms the integral

$E_n = \frac{1}{n!}\int_c^x (x-t)^n . f^{(n+1)}(t)\, dt$ to the form $E_n = \frac{(x-c)^{n+1}}{n!}\int_c^x u^n . f^{(n+1)}(x+(c-x)u)\, dt$

For t = x+(c+x)u $u = \frac{t-x}{c-x}$

dt = 0+(c-x) du

$du = \frac{dt}{c-x}$

t	c	x
u	1	0

$E_n = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\, dt$ becomes

$E_n = \frac{1}{n!}\int_1^0 (x-c)^n u^n . f^{(n+1)}(x+(c+x)u)\,(c-x)dt$

$E_n = \frac{1}{n!}\int_1^0 -(x-c)^n u^n . f^{(n+1)}(x+(c+x)u)dt$

$E_n = \frac{(x-c)^{n+1}}{n!}\int_c^x u^n . f^{(n+1)}(x+(c-x)u)\, dt$

Theorem 4.41: [Bernstein Theorem]

Assume 'f' and all its derivatives are non-negative on a compact interval [b, b+r]. Then if b≤x< b+r, the Taylor's series $\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k$ converges to f(x).

Proof:

Assume 's' and all its derivatives are non-negative on a compact interval [b, b+r].

To prove: If b≤x< b+r, $\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k$ converges to f(x).

By a translation, we can assume that b=0

∴ b≤x< b+r becomes 0≤x< r

The result is trivial for x= 0

Assume that 0<x<r

We know that the Taylor's Formula with remainder

$f(x) = \sum_{k=0}^{n} \frac{f^{(k))}(0)}{k!}(x)^k + E_n(x)$ ....(1)

where $E_n = \frac{1}{n!}\int_0^x (x-t)^n . f^{(n+1)}(t)\, dt$ ....(2)

We will prove that the error term satisfied the inequality $0 \le E_n \le (\frac{x}{r})^{n+1}.f$®

Put t=x-xu in (2), we get

$E_n = \frac{1}{n!}\int_0^x (x-t)^n . f^{(n+1)}(t)\, dt$ t =x-xu

t	o	x
u	1	0

$E_n = \frac{1}{n!}\int_1^0 (xu)^n . f^{(n+1)}(x-xu)\,(-xdu)$

$E_n = \frac{(x)^{n+1}}{n!}\int_0^1 u^n . f^{(n+1)}(x-xu)\, du$ $\forall$ $x \in [0,r]$

If $x \neq 0$, let $F_n(x) = \frac{E_n(x)}{x^{n+1}}$

$F_n(x) = \frac{(x)^{n+1}}{n!}\int_0^1 u^n . f^{(n+1)}(x-xu)\, du$

Now, $f^{(n+1)}(x-xu) = f^{(n+1)}\, x(1-u)$

$\le f^{(n+1)}\, r(1-u)$ if $0 \le u \le 1$

[Since f^{n+1} is monotonic increasing on [0, r] & its derivative non–negative]

$\therefore \frac{1}{n!}\int_0^1 u^n . f^{(n+1)}(x-xu) \le \frac{1}{n!}\int_0^1 u^n . f^{(n+1)}(r-ru)\, du$

$F_n(x) \le F_n(r)$ if $0 < x \le r$

(i.e.) $\frac{E_n(x)}{x^{n+1}} \le \frac{E_n(r)}{x^{r+1}}$

(i.e.) $E_n(x) \le \left(\frac{x}{r}\right)^{n+1} E_n(r)$ ....(3)

Put x=r in (1) we get

f(r)= $\sum_{k=0}^{n} \frac{f^{(k))}(0)}{k!} r^k$ + $E_n(x)$

f(r)≥ $E_n(r)$

∴ from equation (3) ⇒ $E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} f(r)$

Now, Clearly $\left(\frac{x}{r}\right)^{n+1}$ tends to 0 if 0<x<r

$\left(\frac{x}{r}\right)^{n+1}$. f(r) → 0 as $n \to \infty$ (by Comparison Test)

from equation (1) ⇒ f(x) $\lim_{n\to\infty} \frac{f^{(k)}(0)}{k!} x^k$

Hence the Taylor series $\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x-b)^k$ converges to f(x).

Unit V

Sequences of Functions – Pointwise convergence of sequences of functions -Examples of sequences of real - valued functions - Uniform convergence and continuity -Cauchy condition for uniform convergence - Uniform convergence of infinite series of functions - Riemann - Stieltjes integration – Non-uniform Convergence and Term-by-term Integration - Uniform convergence and differentiation - Sufficient condition for uniform convergence of a series - Mean convergence.

Sequences of functions

Point wise Convergence of Sequences of Functions

Definition 5.1:

Let S be the set. The function f defined by the equation f(x) = $\lim_{n\to\infty} f_n(x)$ if $x \in S$ is called the limit function of the sequence $\{f_n\}$, and we say that $\{f_n\}$ converges pointwise to 'f' on the set 'S'.

Examples of Sequences of Real-Valued Functions

Example 5.2:

A sequence of continuous functions with a discontinuous limit function.

Let $f_n(x) = \frac{x^{2n}}{(1+x^{2n})}$ if $x \in R$, n=1,2,….

Here $\lim_{n\to\infty} f_n(x)$ exists for all $x \in R$,

The limit function f(x) is given by $f(x) = \begin{cases} 0 & if\ |x| < 1 \\ 1/2 & if\ |x| = 1 \\ 1 & if\ |x| > 1 \end{cases}$

Each f_n is continuous on R, but f is discontinuous at x=1 & x= -1.

Example 5.3:

A Sequence of functions for which $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$

Let $f_n(x) = n^2 x\,(1-n)^n$ if $x \in R$, n=1, 2,...

If 0≤x≤1, then the limit

$f(x) = \lim_{n\to\infty} f_n(x) = 0$

$\therefore \int_0^1 f(x)\,dx = 0$ (i.e.) $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$

Now,

$\int_0^1 f_n(x)dx = n^2\int_0^1 x(1-x)^n\,dx$ [x=1-t ⇒t=1-x & dx = - dt ⇒

x	0	1
t	1	0

$= n^2\int_1^0 (1-t)t^n\,{-dt}$

$= n^2\int_0^1 (1-t)t^n\,dt$

$= n^2\int_0^1 t^n-t^{n+1}\,dt$

$= n^2\left[\frac{t^{n+1}}{n+1}-\frac{t^{n+2}}{n+2}\right]_0^1$

$= n^2\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

$= n^2\left[\frac{n+2-n-1}{(n+1)(n+2)}\right]$

$= \frac{n^2}{(n+1)(n+2)}$

$\therefore \int_0^1 f_n(x)dx = \frac{n^2}{(n+1)(n+2)}$

$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\frac{n^2}{(n+1)(n+2)} = 1$

(i.e.) $\lim_{n\to\infty}\int_0^1 f_n(x)dx = 1$

Hence $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

In other words, the limit of the integral is not equal to the integral of the limit function

∴ The operations of "limit" and integration" cannot always be interchanged.

Example 5.4:

A sequence of differentiable function $\{f_n\}$ with limit 'o' for which $\{f_n'\}$ diverges.

Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ if $x \in R$, n=1,2,……

Then $f(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0$

Now, $f_n'(x) = \frac{1}{\sqrt{n}}(\cos nx).\ n$

$= \sqrt{n}(\cos nx)$

$\therefore f_n'(x) = \sqrt{n}(\cos nx)$

$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sqrt{n}(\cos nx) = \infty$

$\therefore \lim_{n\to\infty} f_n'(x)$ does not exist for any 'x'.

(i.e.) $\{f_n'\} \to \infty$

Hence $\{f_n\} \to 0$ But $\{f_n'\} \to \infty$.

Definition of Uniform convergence

Definition 5.5:

A sequence of functions $\{f_n\}$ is said to converge pointwise to 'f' on a set 'S' if

for all $\varepsilon>0$, for all $x \in S$, there exist N(depending on both x & ε)such that $n > N \Rightarrow |f_n(x)-f(x)|<\varepsilon$

Definition 5.6:

A sequence of functions $\{f_n\}$ is said to converge uniformly to 'f' on a set 'S' if

for all $\varepsilon>0$, there exist N(depending only on ε)such that $n > N$

$\Rightarrow |f_n(x)-f(x)|<\varepsilon, \forall x \in S$

We denote this symbolically by writing $f_n \to f$ uniformly on S.

Note:

When each term of the sequence $\{f_n\}$ is real-valued, Then $|f_n(x) - f(x)| < \varepsilon$ becomes

$-\varepsilon < f_n(x) < f(x) + \varepsilon$

(i.e.)for all $\varepsilon>0$, there exist N(depending only on ε)such that $n> N$

$\Rightarrow f(x)-\varepsilon< f_n(x)<f(x) +\varepsilon, \forall\, x\in S$

(i.e.) The entire graph of $f_n = \{(x,y) : y = f_n(x), x\in S\}$ lies within a band of height 2ε situated symmetrically about the graph of 'f'.

Definition 5.7:

A sequence $\{f_n\}$ is said to be uniformly bounded on S if there exists a constant $M>0$. Such that $|f_n(x)\leq M\ \forall\, x\in S\ \&\ \forall\, n$. The number M is called a uniform bound for $\{f_n\}$.

Note (i):

Assume that $f_n\to f$ uniformly on S and that each f_n is bounded on S. Then $\{f_n\}$ is uniformly bounded on S

Given $f_n\to f$ uniformly on S

$\forall\, \varepsilon > 0$, there exist N such that $n>N \Rightarrow |f_n(x) - f(x)|< \varepsilon$

$\Rightarrow ||f_n(x)|-|f(x)||<|f_n(x)-f(x)|<\varepsilon\ \ \forall\, x\in S$

$\Rightarrow |f_n(x)|-|f(x)|<||f_n(x)-f(x)|| <\varepsilon\ \ \forall\, x\in S$

$\Rightarrow |f_n(x)|-|f(x)|<\varepsilon$

$\Rightarrow |f_n(x)|-|f(x)|<\varepsilon$ ....(1)

$|f(x)|-|f_n(x)|<\varepsilon$ ....(2)

Given, f_n is bounded on S

$\Rightarrow$there exist $M_1>0$ such that $|f_n(x)<M_1\ \forall\, x\in S$

Equation (2)$\Rightarrow |f(x)|< M_1 + \varepsilon$

Equation (1)$\Rightarrow |f_n(x)|<|f(x)|+\varepsilon < (M_1+\varepsilon)+\varepsilon = M_1+2\varepsilon =M$ where $M=M_1+2\varepsilon$

(i.e.) $|fn(x)|<M\ \forall\, x\in S\ \forall\, n$

$\{f_n\}$ is uniformly bounded on S.

Note (ii):

If $f_n \to f$ uniformly on S & each f_n is bounded, then $\{f_n\}$ need not be uniformly converges

Note (iii):

If 'c' is an accumulation point of S, then $\lim_{x\to c} \lim_{n\to\infty} f(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$

Uniform Convergence and Continuity

Theorem 5.8:

Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at a point 'c' of S, then the limit function 'f' is also continuous at c.

Proof:

Assume that $f_n \to f$ uniformly on s

$\Rightarrow$ for all $\varepsilon>0$, there exist N such that $n> N \Rightarrow |f_n(x)-f(x)|<\varepsilon/3$ for all $x\in S$

Case(i): 'c' is an isolated point on 'S'

By the definition of isolation point, we get

There exist $\delta>0$ such that $(c-\delta,c+\delta)\cap S = \{c\}$

Let $\varepsilon>0$ be given

Let $|x-c|<\delta$

To prove that $|f(x)-f(c)|< \varepsilon$

Now, $|x-c|<\delta$

$\Rightarrow -\delta<x-c<\delta$

$\Rightarrow c-\delta<x<c+\delta$

$\Rightarrow x\in(c-\delta,c+\delta)$

Also $x\in S$

$\therefore x \in (c-\delta, c+\delta) \cap S = \{c\}$

$\therefore x = c$

Now, $|f(x)-f(c)|=|f(c) - f(c)|= 0 < \varepsilon$.

$\therefore$ 'f' is continuous at 'c'.

Case (ii):

'c' is an accumulation point on 'S'

Given: Each f_n is continuous at 'c'

$\therefore f_n$ is continuous at 'c'.

There exist a neighbourhood B(c) such that $x \in B(c) \cap S$ implies

$|x-c|<\delta \Rightarrow |f_N(x)-f_N(c)|<\varepsilon/3$ …….(2)

Now, $|f(x) - f(c)| = |f(x)-f_N(x)+f_N(x)-f_N(c)+f_N(c) - f(c)|$

$\leq |f(x)-f_N(x)|+|f_N(x)-f_N(c)|+|f_N(c) - f(c)|$

$\leq \varepsilon/3+\varepsilon/3 +\varepsilon/3$

$= \varepsilon$ (by equation (1) & (2))

$\therefore |f(x) - f(c)|< \varepsilon$ if $x \in B(c) \cap S$

$\therefore |x-c|<\delta \Rightarrow |f(x)-f(c)| < \varepsilon$

$\therefore f$ is continuous at c.

Note: Uniform convergence of $\{f_n\}$ is sufficient but not necessary to transmit continuity from the individual terms to the limit function.

The Cauchy Condition for Uniform Convergence

Theorem 5.9:

Let $\{f_n\}$ be a sequence of functions defined on a set S. There exists a function 'f' such that $f_n \rightarrow f$ uniformly on S if and only if the following condition (called the Cauchy Condition) is satisfied: For every $\varepsilon > 0$ there exists an N such that $m>N$ & $n>N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon \ \forall x \in S$.

Proof:

Let $\{f_n\}$ be a sequence of functions defined on a set 'S'.

Suppose there exist 'f' such that: $f_n \rightarrow f$ uniformly on S

$\Rightarrow$for all $\varepsilon>0$, there exist N such that $n> N \Rightarrow |f_n(x)-f(x)|<\varepsilon/2$ for all $x\in S$

Now, $|f_m(x) - f_n(x)| = |f_m(x) - f(x)+f(x)- f_n(x)|$

$\leq |f_m(x) - f(x)|+|f(x)- f_n(x)| < \varepsilon$

$< \varepsilon/2+\varepsilon/2 = \varepsilon$

Hence for all $\varepsilon>0$, there exist N such that

$m>N$ & $n> N \Rightarrow |f_m(x)-f_n(x)|<\varepsilon$ for all $x\in S$

Conversely,

Suppose the Cauchy Condition is satisfied

(i.e.)for every $\varepsilon>0$ there exist N such that

$m>N$ & $n>N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon \ \forall\ x\in S$ ……….(1)

To prove that, a function 'f' such that $f_n \rightarrow f$ uniformly on S

From (1), we get

For each $x\in S$, the sequence $\{f_n(x)\}$ converges.

To prove: $\lim_{n\to\infty} f_n(x) = f(x)$ if $x\in S$

Let $\varepsilon>0$ be given

Choose N so that

$n>N\ |f_n(x) - f_{n+k}(x)| < \varepsilon/2\ \forall\ K=1,2,3\ldots..$ & $\forall\ x\in S$

$\lim_{n\to\infty} |f_n(x) - f_{n+k}(x) = |f_n(x) - f(x)|\leq\varepsilon/2 <\varepsilon$

Hence $n>N \Rightarrow |f_n(x) - f(x)| < \varepsilon\ \forall\ x \in S$.

$\therefore f_n \rightarrow f$ uniformly on s

Note:

- Pointwise and uniform convergence can be formulated in the more general setting of metric spaces. (i.e.) if f_n & f are functions from a non-empty set S to a metric space (T, d_T), we say that $f_n \to f$ uniformly on S, If for every $\varepsilon>0$, there exist N (depending only on ε) such that $n \geq N \Rightarrow d_T(f_n(x), f(x)) < \varepsilon$ for all $x \in S$
- Theorem 5.8 & 5.9 is valid if S is a metric space also.

Example 5.10:

Consider the metric space (B(S),d) of all bounded real-valued functions on a non-empty set S, with metric $d(f,g)= \|f-g\|$ where $\|f\|=\sup_{x\in S} |f(x)|$.

Then $f_n \to f$ in (B(S), d) ⇔ $f_n \to f$ uniformly on S.

(i.e.) ordinary convergence in a metric space (B(S), d) ⇔ Uniform convergence on S

Proof:

Suppose $f_n \to f$ in (B(S), d)

$\Rightarrow \forall\, \varepsilon>0, \forall\, x\in S$, there exist N (depending on both x & ε) such that

$n>N \Rightarrow \|f_n(x) - f(x)\| < \varepsilon$

(i.e.) $n>N \Rightarrow \sup_{x\in S} |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow \forall\, \varepsilon>0$, there exist N (depending only on ε) such that

$n>N \Rightarrow \sup_{x\in S} |f_n(x) - f(x)| < \varepsilon \quad \forall\, x\in S$ ……(1)

We know that $|f_n(x) - f(x)| \leq \sup_{x\in S} |f_n(x) - f(x)|$ (by 1)

$< \varepsilon$

$\therefore \forall\, \varepsilon >0$, there exist N (depending only on ε)such that

$n>N \Rightarrow |f_n(x)-f(x)|<\varepsilon \quad \forall\, x\in S$

Hence $f_n \to f$ uniformly on S.

Similarly, we can prove the converse part also.

Uniform Convergence of Infinite Series of Functions

Definition 5.11:

Given a sequence $\{f_n\}$ of functions defined on a set S.

For each x in S, let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ (n=1,2,…..)

If there exists a function f such that in $S_n \rightarrow f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S, and write $\sum_{n=1}^{\infty} f_n = f(x)$ (uniformly on s)

Theorem 5.12:

[Cauchy Condition for Uniform Convergence of Series]

The infinite series $\sum f_n(x)$ converges uniformly on S, if and only if for every ε>0 there is an N such that n>N ⇒ $\left|\sum_{k=n+1}^{n+p} f_k(x) < \varepsilon\right|$, for each p=1, 2,... and every x in S.

Proof:

Let $\{f_n\}$ be a sequence of functions defined on S.

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ (n=1,2,...) $\forall x \in S$

Given, $\sum f_n(x)$ converges uniformly on s

⇒there exist a function f such that $S_n \rightarrow f$ uniformly on S

By the Cauchy Condition for Uniform Convergence of the sequence

Theorem 5.9, we get ∀ ε>0, there exist N such that n>N

$\Rightarrow |S_{n+p}(x) - S_n(x)| < \varepsilon$, p = 1,2,3,…….&$\forall x \in S$

$\Rightarrow |\sum_{k=1}^{n+p} f_k(x) - \sum_{k=1}^{n} f_k(x)| < \varepsilon$, p = 1,2,3,…….&$\forall x \in S$

∴ n>N $\Rightarrow |\sum_{k=1}^{n+p} f_k(x)| < \varepsilon$, p = 1,2,3,…….&$\forall x \in S$

Conversely,

Assume that ∀ ε>0 there exist N such that

$n>N \Rightarrow |\sum_{k=1}^{n+p} f_k(x)| < \varepsilon$, p = 1,2,3,…….&$\forall$ $x \in S$

$\Rightarrow |\sum_{k=1}^{n+p} f_k(x) - \sum_{k=1}^{n} f_k(x)| < \varepsilon$,

$\Rightarrow |S_{n+p}(x) - S_n(x)| < \varepsilon$, p = 1,2,3,…….&$\forall$ $x \in S$

Again by Cauchy condition for Uniform Convergence (Theorem 5.9), we get

$S_n \to f$ uniformly on S

$\Rightarrow \Sigma f_n(x)$ converges uniformly on S

Theorem 5.13 [Weierstrass M-test]

Let $\{M_n\}$ be a sequence of non-negative numbers Such that $0 \leq |f_n(x)| \leq M_n$ for n=1,2, & $\forall x \in S$. Then $\Sigma f_n(x)$ converges uniformly on S if M_n converges.

Proof:

Let $\{f_n\}$ be a sequence of functions defined on S

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ (n=1,2,….) $\forall$ $x \in S$

Let $\{M_n\}$ be a sequence of non-negative numbers such that

$0 \leq | f_n(x)] \leq M_n$ for n=1, 2 ,... & $\forall$ $x \in S$

Given: $\sum M_n$ converges

To prove: $\sum f_n(x)$ converges uniformly on S

(i.e.) To prove that there exist a function f , $S_n \to f$ uniformly on S

$\therefore \sum M_n$ converges & by Cauchy Condition for Series

$\forall \varepsilon>0$ there exist N such that $n>N \Rightarrow |M_{n+1}+M_{n+2}+ \ldots\ldots\ldots +M_{n+p}| < \varepsilon$ for p = 1,2,….

$\Rightarrow |\sum_{k=1}^{n+p} M_k(x)| < \varepsilon$ for p = 1,2,….. …. (1)

Now, Given: $0 \leq |f_n(x)| \leq M_n$

$\Rightarrow |\sum_{k=1}^{n+p} f_k(x)| \leq |\sum_{k=1}^{n+p} M_k|$

$< \varepsilon$ (by(1))

$\therefore|\sum_{k=1}^{n+p} f_k(x)| < \varepsilon$, for p = 1,2,……& $\forall\ x \in S$

By Cauchy Condition for Uniform Convergence of series (Theorem 5.12), we get

$\sum f_n(x)$ converges uniformly on s

Theorem 5.14:

Assume that $\sum f_n(x) = f(x)$ (uniformly on s). If each f_n is continuous at a point x_0 of S, then f is also continuous at x_0.

Proof:

Let $\{f_n\}$ be a sequence of functions defined on S

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ (n=1,2,….) & $\forall\ x \in S$

Given: $f_n(x) = f(x)$ (uniformly on s) $\Rightarrow S_n \to f$ uniformly on S

$\Rightarrow \forall\ \varepsilon>0$, there exist N such that n> N = $|S_n(x) - f(x)| < \varepsilon/3$ ……(1)

Given: Each f_n is continuous at x_0

$\Rightarrow S_n$ is continuous at x_0

$\therefore S_N$ is continuous at x_0

$\Rightarrow$there exist a neighbourhood $B(x_0)$

$x \in B(x_0) \cap S \Rightarrow |S_N(x) - S_N(x_0)| < \varepsilon/3$ …………(2)

If $x \in B(x_0) \cap S$, then $|f(x) - f(x_0)| = |f(x) - S_N(x) + S_N(x) - S_N(x_0) + S_N(x_0) - f(x_0)|$

$\leq |f(x) - S_N(x)| + |S_N(x) - S_N(x_0)| + |S_N(x_0) - f(x_0)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ (by equation (1) +(2))

$< \varepsilon$

(i.e.) $|f(x) = f(x_0)| < \varepsilon$

Hence f is continuous at x_0

Non-Uniformly Convergent Sequences that can be Integrated Term by Term:

Example 5.15:

Let $f_n(x) = x^n$ if $o \le x \le 1$

Then $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & if\ o < x < 1 \\ 1 & if\ x = 1 \end{cases}$

$\therefore f_n(x)$ is a sequence of continuous functions with discontinuous limit

$\Rightarrow$ The convergence of $f_n(x)$ is not uniform on [0,1]

Now,

$\int_0^1 f_n(x)dx = \int_0^1 x^n dx = \frac{x^{n+1}}{n+1} = \frac{1}{n+1}$

$\Rightarrow \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\therefore$The sequence $f_n(x)$ is not uniformly convergent on [0,1] But this sequence $f_n(x)$ is uniformly convergent on every closed sub interval of [0,1] not containing 1.

Definition 5.16:

A sequence of functions {fn} is said to be boundedly convergent on T if $\{ f_n \}$ is pointwise convergent and uniformly bounded on T

Theorem 5.17:

Let $\{f_n\}$ be a boundedly convergent sequence on [a, b]. Assume that each $f_n \in R$ on [a, b], and that the limit function $f \in R$ on [a, b]. Assume also that there is a partition P of [a, b], say P = $\{x_0, x_1, x_2, \ldots x_m\}$, Such that, on every sub interval [c, d] not containing any of the points x_k, the sequence $\{f_n\}$ converges uniformly to f. Then we have $\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)dt$ $= \int_a^b f(t)\,dt$

Proof:

Let $\{ f_n\}$ be a boundedly convergent sequence on [a, b]

$\Rightarrow \{ f_n \}$ is point wise convergent and uniformly bounded on [a, b].

Assume each $f_n \in$ R on [a, b] and $f \in$R on [a, b]

Let p = { $x_0, x_1, x_2, \ldots . x_m$} $\in \mathcal{P}$[a,b] such that every subinterval [c,d] not containing any of the points x_k, the sequence {f_n} converges uniformly to 'f'.

To prove that $\lim_{n\to\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim_{n\to\infty} f_n(t)\, dt = \int_a^b f(t)\, dt$

∵ f is bounded and {f_n} is uniformly bounded on [a, b]

There exist M such that $|f(x)| \leq M \ \forall\ x \in [a, b]$ & $|f_n(x)| \leq M \ \forall\ x \in [a, b]$

and $\forall\ n \geq 1$ …. ….. (1)

given ε>0 such that 2ε ≤ ‖p‖

Let h=ε/2M, where m = Number of sub intervals of P

Consider a new partition p' of [a, b] given by

p= {x_0, x_{0+h},x_{1-h},x_{1+h}….x_{m-1}-h,x_{m-1},x_{m-1}+h,x_m-h,x_m}

Now, $|f - f_n| = |f+(-f_n)| \leq |f| + |-f_n| = |f| + |f_n| \leq M+M$ (by 1)

$|f-f_n| \leq 2M$ ……. (2)

Now, $f \in R$ & $f_n \in R$ on [a, b]

$|f-f_n| \in R$ on [a, b]

∴The sum of the integrals of $|f-f_n|$ taken over the intervals

[x_0, x_{0+h}],[x_{1-h},x_{1+h}],….[x_{m-1}-h,x_{m-1}+h],[x_m-h, x_m] is

$$\int_{x_0}^{x_0+h} |f - f_n|\, dx + \int_{x_1}^{x_1+h} |f - f_n|\, dx + \ldots\ldots + \int_{x_{m-1}}^{x_{m-1}+h} |f - f_n|\, dx$$

$$+ \int_{x_{m-h}}^{x_m} |f - f_n|\, dx$$

≤2M {[x_0, x_{0+h}],[x_{1-h},x_{1+h}]….[x_{m-1}-h,x_{m-1}+h],[x_m-h, x_m]}

=2M {h+2h+2h+…….2h+h}

=2M [2h+2h+……+2h] (m times)

=2M(2h).m

$=2M(2m)\varepsilon/2m$

$=2M\varepsilon$

(i.e.) $\int_{x_0}^{x_0+h}|f-f_n|\,dx+\int_{x_1-h}^{x_1+h}|f-f_n|\,dx+\ldots\ldots+\int_{x_m-h}^{x_m}|f-f_n|\,dx \le 2M\varepsilon$ …(3)

The remaining portion of [a, b] (say S) is the union of finite number of closed intervals, in each of which $\{f_n\}$ is uniformly converges to 'f'.

(i.e.), $f_n \to f$ uniformly on S

$\Rightarrow$there exist an integer N such that $n\ge N \Rightarrow |f(x)-f_n(x)| < \varepsilon\ \forall x\in S$

The sum of the integrals of $|f-f_n|$ over the intervals of S is

$$\int_{x_0+h}^{x_1+h}|f-f_n|\,dx+\int_{x_1-h}^{x_2+h}|f-f_n|\,dx+\ldots\ldots+\int_{x_{m-2}}^{x_{m-1}-h}|f-f_n|\,dx$$

$$+\int_{x_{m-1}+h}^{x_m-h}|f-f_n|\,dx$$

$<\varepsilon\{[x_1-h-x_0-h]+[x_2-h-x_1-h]+\ldots..+[x_{m-1}-h-x_{m-2}-h]+[x_m-h-x_{m-1}-h]\}$

$= \varepsilon\ \{(-x_0-2h)+(-2h)+\ldots\ldots+(-2h)+(x_m-2h)\}$

$= \varepsilon\ \{x_m-x_0-(2h+2h+\ldots..2h)$ (m times)}

$= \varepsilon\ \{b-a-2h(m)\}$

$= \varepsilon\ \{b-a-2m.\varepsilon/2m\}$

$= \varepsilon\ [(b-a)-\varepsilon\}$

$= \varepsilon\ (b-a) - \varepsilon^2$

$\le \varepsilon\ (b-a)$

(i.e.) $\int_{x_0+h}^{x_1+h}|f-f_n|\,dx+\int_{x_1-h}^{x_2+h}|f-f_n|\,dx+\ldots\ldots+\int_{x_{m-1}+h}^{x_m-h}|f-f_n|\,dx \le \varepsilon(b-a)$

…… (4)

From (3) and (4) we get,

$\int_a^b |f(x) - f_n(x)|dx \le 2M\varepsilon + \varepsilon(b-a)$

$= \varepsilon\ (2M+(b-a))$

(i.e.) $\int_a^b |f(x) - f_n(x)| dx \leq = \varepsilon\ (2M+(b-a))$ where ever $n \geq N$

$\therefore \int_a^b f(x)\ dx \Rightarrow \int_a^b f(x)\ dx$ as $n \to \infty$

(i.e.) $\lim_{n\to\infty} \int_a^b f_n(x)\ dx \Rightarrow \int_a^b f(x)\ dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx$

Theorem 5.18:(Arzela)

Assume that $\{f_n\}$ is boundedly convergent on [a, b] and suppose each f_n is Riemann-integrable on [a, b]. Assume also that the limit function 'f' is Riemann-integrable on [a, b]. Then $\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx = \int_a^b f(x)\ dx$

Example 5.19:

A boundedly convergent sequence $\{f_n\}$ of Riemann- Integrable functions whose limit is not Riemann- Integrable.

(i.e.), $\lim_{n\to\infty} \int_a^b f_n(x)\ dx \neq \int_a^b \lim_{n\to\infty} f_n(x)\ dx = \int_a^b f(x)\ dx$

Let $\{r_1, r_2 \ldots\ldots\}$ be the set of rational numbers in [0,1]

Define $f_n(x) = \begin{cases} 1 & if\ x\ = r_k \\ 0 & otherwise \end{cases}$ $\forall$ k=1,2,……n

$f(x) = \begin{cases} 1\ \ if\ x\ is\ rational \\ 0\ if\ x\ is\ irrational \end{cases}$

Clearly, $\therefore \int_0^1 f_n(x)\ dx = 0\ \forall\ n$

Here 'f_n' converges point wise to f

∴ Each f_n has only finitely many points of discontinuity

∴ Each f_n is Riemann- Integrable.

But $U(P, f) = \sum M_k(f) \Delta x_k$

$= \sum \sup(f(x)).\ \Delta x_k$

$= \sum 1.\ \Delta x_k$

=b-a

= 1-0 = 1

∴ U(P, f) = 1 ⇒ $\int_{a}^{-b} f(x)\,dx$

Similarly, L(P, f) = $\sum M_k (f)\Delta x_k$

=$\sum \inf (f(x).\ \Delta x_k$

= $\sum 0.\ \Delta x_k$

=0

⇒ $\int_{-a}^{b} f(x)\,dx$

∴$\int_{-a}^{b} f(x)\,dx \neq \int_{-a}^{b} f(x)\,dx$.

∴The limit function f is not Riemann- Integrable.

Uniform Convergence and Differentiation

Note

- $\{f_n\}$ converges uniformly on R. Then $\{f_n'\}$ need not converge (even pointwise) on R
- If $f_n \to f$ uniformly on [a,b] & if $f_{n'}$ exists for each n then f' exists & $f'_n \to f'$ uniformly on [a b] need not be true.

Theorem 5.20:

Assume that each term of $\{f_n\}$ is a real-valued. function having a finite derivative at each point of an open interval (a, b) Assume that for at least one point x_0 in (a, b) the sequence $\{f_n (x_0)\}$ converges. Assume further that there exists a function g such that $f'_n \to g$ uniformly on (a,b). Then

a) There exists a function f such that $f_n \to f$ uniformly on (a, b).

b) For each x in (a, b) the derivative f'(x) exists and equals g(x).

Proof:

Assume that each term of $\{f_n\}$ is a real-valued function having a finite derivative at each point of (a, b)

Given: Atleast one point $x_0 \in(a,b)$, $\{f_n(x_0)\}$ converges. …… (1)

Given: the exist 'g' such that : $f' \to g$ uniformly on (a,b) …….(2)

(a) there exist as $f_n \to f$ uniformly on (a, b)

Assume that $c \in (a,b)$

Define a new sequence $\{g_n\}$ as follows.

$$g_n(x) = \begin{cases} \frac{f_n(x)-f_n(c)}{x-c} & if\ x \neq c \\ f_n' & if\ x = c \end{cases} \quad ....(3)$$

The sequence $\{g_n\}$ so formed depends on the choice of 'c'

(3) $\Rightarrow g_n$ (c)-f_n'(c)

(2) $\Rightarrow\{f_n'(c)\}$ converges

(i.e.) $\{g_n(c)\}$ converges

Claim: $\{g_n\}$ converges uniformly on (a,b)

If $x \neq c$, then, $g_n(x) - g_m(x) = \frac{f_n(x) - f_n(c)}{x-c} - \frac{f_m(x)-f_m(c)}{x-c}$

$\Rightarrow g_n(x) - g_m(x) = \frac{[f_n(x)-f_m(c)]-[f_n(x)-f_m(c)]}{x-c}$

Let $h(x) = f_n(x) - f_m(x)$

$\therefore g_n(x) - g_m(x) = \frac{h(x)-h(c)}{x-c}$ …….(4)

Now, $h(x) = f_n(x)-f_m(x)$

$\Rightarrow h'(x) = f'_n(x) - f'_m(x)$ & $h'(x)$ exists $\forall\ x \in(a,b)$ …….(5)

By Mean-Value Theorem,

There exist a point $x_1 \in(x, c)$

such that $h(x) - h(c) = h'(x_1)(x-c)$

$\Rightarrow h'(x_1) = \frac{h(x)-h(c)}{x-c}$

$f'_n(x) - f'_m(x) = g_n(x) - g_m(x)$ ....(6) (by (4) and (5))

Given $f'_n \to g$ uniformly on (a,b)

$\Rightarrow$ given $\varepsilon>0$, there exist N such that $\forall$ n>N $\Rightarrow |f'_n(x)-g(x)| < \varepsilon/2 \ \forall x \in (a,b)$

Let n, m>N.

Then $| g_n(x) - g_m(x) | = | f'_n(x_1) - f'_m(x_1)|$ (by equation (1))

$= |f'_n(x_1) - g(x_1) + g(x_1) - f'_m(x_1)|$

$\leq |f'_n(x_1) - g(x_1)| + | f'_m(x_1) - g(x_1)|$

$< \varepsilon/2 + \varepsilon/2$

$= \varepsilon$

(i.e.) $|g_n(x) - g_m(x)| < \varepsilon$

$\therefore \{g_n\}$ converges uniformly on (a,b)

Now, to prove: $\{f_n\}$ converges uniformly on (a,b)

Let us form the particular sequence $\{g_n\}$ corresponding to the special point $c=x_0$ for which $\{f_n(x)\}$ is assumed to converge.

(3) $\Rightarrow g_n(x) = \frac{f_n(x)-f_n(x_0)}{x-x_0}$

$\Rightarrow f_n(x) = f_n(x_0) + (x-x_0)\, g_n(x) \ \forall x \in (a,b)$

$\therefore f_n(x) - f_m(x) = f_n(x_0) + (x-x_0)\, g_n(x) - f_m(x_0) - (x-x_0) g_m(x)$

$\Rightarrow f_n(x) - f_m(x) = [f_n(x_0) - f_m(x_0)] + (x-x_0)[g_n(x) - g_m(x)]$ ....(7)

Now, $\{g_n\}$ converges uniformly on (a,b) & $\{f_n(x_0\}$ converges

$\Rightarrow$given $\varepsilon>0$, choose N such that for n,m> N

$|g_n(x) - g_m(x)| < \varepsilon/2|x-x_0|$ & $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$

∴ equation (7) ⇒ $|f_n(x) - f_m(x)| < \varepsilon/2 + |x-x_0|.\varepsilon/2|x-x_0|$

$= \varepsilon/2 + \varepsilon/2 = \varepsilon$

∴ $\{f_n\}$ converges uniformly on (a, b)

b) Let 'c' be an arbitrary point in (a, b)

Let $G(x) = \lim_{n\to\infty} Gn(x)$

Given f'_n exists

Equation (3) $\Rightarrow \lim_{x\to c} g_n(x) = g_n(c)$

(i.e.), Each g_n is continuous at c

We have $\{g_n\}$ converges uniformly on (a, b)

$g_n \to g$ converges uniformly on (a, b)

∴G is also continuous at 'c'

(i.e.), $\lim_{x\to c} G(x) = G(c)$.

For x≠c, we have,

(i.e.), $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f_n(x)-f_n(c)}{x-c} = \frac{f(x)-f(c)}{x-c}$

∴ $\lim_{x\to c} G(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

$G(c) = f'(C)$ ……….(8)

But also

$G(c) = \lim_{n\to\infty} g_n(c)$

$= \lim_{n\to\infty} f'_n(c)$

$= g(c)$ [by equation (3)] [∵ $f'_n \to g$ uniformly on (a,b)

(i.e.) $G(c) = f'(c)$ ………..(9)

From equation (8) & (9) we get, $f'(c) = g(c)$

∴ c is arbitrary, we get

$f'(x) = g(x)$

Theorem 5.21:

Assume that each f_n is a real-valued function defined on (a, b) such that the derivative $f_n'(x)$ exists for each x in (a,b). Assume that, for at least one point x_0 in (a, b), the series $\sum f_n(x_0)$ converges. Assume further that there exists a function g such that $\sum f_n'(x) = g(x)$ (uniformly on (a, b)). Then

a) There exists a function f such that $\sum f_n(x) = f(x)$ (uniformly on (a, b)

b) If $x \in (a,b)$, the derivative $f'(x)$ exists and equals $\sum f_n'(x)$

Proof:

Define $s_n'(x) = \sum_{k=1}^{n} f_k'(x) \forall x \in (a,b)$ & $S_n(x_0) = \sum_{k=1}^{n} f_k'(x_0)$

Given $\sum f_n'(x) = g(x)$ (uniformly on (a,b))

$\Rightarrow \{S_n'\} \longrightarrow g(x)$ (uniformly on (a,b)) …………(1)

Given $f_n(x_0)$ converges

$\Rightarrow \{s_n'(x_0)\}$ converges ……….. (2)

By Theorem 9:13 (a) & by (1) & (2)

There exist f such that: $\{S_n\} \longrightarrow f$ uniformly on (a,b)

$\therefore \sum f_n(x) = f(x)$ (uniformly on (a, b))

By Theorem 5.20 (b), & by equation (1) & (2)

For each $x \in (a,b)$, the derivative $f'(x)$ exists

and equal to g(x)

(i.e.), $f'(x) = g(x) = \sum f_n'(x)$

Sufficient conditions for Uniform Convergence of a Series

Theorem 5.22: [Dirichlet's Test for Uniform Convergence]

Let $F_n(x)$ denote the n^{th} partial sum of the series $\sum f_n(x)$, where each f_n is a complex-valued function defined on a set S. Assume that $\{F_n\}$ is uniformly bounded on S. Let $\{g_n\}$ be a sequence of real-valued functions such that $G_{n+1}(x) \leq g_n(x)$ for each x in S and for every n=1,2,..., and assume that $G_n \to 0$ uniformly on S. Then the series $\sum f_n(x)\, g_n(x)$ converges uniformly on S.

Proof:

Let each f_n be a complex-valued function defined on S

Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$

Assume that $\{F_n\}$ is uniformly bounded on S

There exist M>0 such that

$|F_n(x)| \leq M$ for all $x \in S$ & for all n ……….(1)

Let $\{g_n\}$ be a sequence of real-valued function such that

$G_{n+1}(x) \leq g_n(x)$ for all $x \in S$ & for all n=1,2,….

Assume that $g_n \to 0$ uniformly on S

$\Rightarrow$ given $\varepsilon>0$ there exist N such that

$n>N \Rightarrow |g_n(x)-0| < \varepsilon/2M$ for all $x \in S$

$\Rightarrow |g_n(x)| < \varepsilon/2M$ for all $x \in S$

To prove that: $\Sigma f_n(x)\, g_n(x)$ converges uniformly on S

Let $S_n(x) = \sum_{k=1}^{n} f_k(x) g_k(x)$

$= \sum_{k=1}^{n} [F_k(x) - F_{k-1}(x)]\, g_k(x)$

$= \sum_{k=1}^{n} F_k(x) g_k(x) - \sum_{k=1}^{n} F_{k-1}(x) g_k(x)$

$= \sum_{k=1}^{n} F_k(x) g_k(x) - \sum_{k=1}^{n} F_k(x) g_{k+1}(x) - F_n(x) G_{n+1}(x) + F_n(x) G_{n+1}(x)$

$=\sum_{k=1}^{n} F_k(x)g_k(x)-\sum_{k=1}^{n} F_k(x)g_{k+1}(x)+F_n(x)G_{n+1}(x)$

$\therefore S_n(x)=\sum_{k=1}^{n} F_k(x)[g_k(x)- g_{k+1}(x)]+F_n(x)G_{n+1}(x)$

If n>m, we can write

$S_n(x) - S_m (x) = \sum_{k=m+1}^{n} F_k(x)[g_k(x)-g_{k+1}(x)]+F_n(x)G_{n+1}(x)-F_m(x)g_{m+1}(x)$

$|S_n(x) - S_m (x)| =| \sum_{k=m+1}^{n} F_k(x)[g_k(x)-g_{k+1}(x)]+F_n(x)G_{n+1}(x)-F_m(x)g_{m+1}(x)|$

$\leq M \mid \sum_{k=m+1}^{n} [g_k(x)-g_{k+1}(x)]+G_{n+1}(x)-g_{m+1}(x)|$ by (1)

$\leq M |g_{m+1}(x)-G_{n+1}(x)+G_{n+1}(x)+g_{m+1}(x)|$

$\leq 2M|g_{m+1}(x)|$

` $< 2M.\frac{\varepsilon}{2M}$

$= \varepsilon$

(i.e.) $|S_n(x) - S_m(x)|< \varepsilon$

$\Rightarrow \{S_n\}$ converges uniformly on S

$\therefore \sum f_n(x)\, g_n(x)$ converges uniformly on S.

Theorem 5.23: Abel's Test for Uniform Convergence]

Let $\{g_n\}$ be a sequence of real-valued functions Such that $g_{n+1}(x) \leq g_n(x)$ for each x in T and for every n = 1, 2,... If $\{g_n\}$ is uniformly bounded on T and if $\sum f_n(x)$ converges uniformly on T, then $\sum f_n(x)\, g_n(x)$ also converges uniformly on T.

Proof:

Let $\{g_n\}$ be a sequence of real-valued functions such that

$g_{n+1}(x)\leq g_n(x)$ for all $x\in T$ & for all n= 1,2,……

Given $\{g_n\}$ is uniformly bounded on T

There exist M>0 such that $|g_n(x)|\leq M$ for all $x \in T$ & $\forall$ n

Let $F_n(x)= \sum_{k=1}^{n} f_k(x)$

Given: $\sum f_n(x)$ converges uniformly on T

$\therefore$ $\{F_n(x)\}$ converges uniformly on T

given $\varepsilon>0$, choose N such that for

$n,m > N \Rightarrow |F_n(x)-F_m(x)|<\varepsilon/M$ ....(2)

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$

To prove: $\sum f_n(x)\, g_n(x)$ converges uniformly on T

(i.e.) To prove: $\{S_n(x)\}$ converges uniformly on T

Now, $S_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$

$\Rightarrow S_n(x)=\sum_{k=1}^{n} [F_k(x)-F_{k-1}(x)]g_k(x)$

$\therefore |S_n(x)-S_m(x)|=|\sum_{k=1}^{n} [F_k(x)-F_{k-1}(x)]g_k(x)-\sum_{k=1}^{n} [F_k(x)-F_{k-1}(x)]g_k(x)|$

$=|\sum_{k=m+1}^{n} [F_k(x)-F_{k-1}(x)]g_k(x)|$

$\leq M\, |\sum_{k=m+1}^{n} [F_k(x)-F_{k-1}(x)]|$ (by (1))

$= M\, |(F_{m+1}(x) - F_m(x) + (F_{m+2}(x)-F_{m+1}(x)) + \ldots\ldots\ldots+(F_n(x)-F_{n-1}(x)|$

$=M\, |F_n(x)-F_m(x)|$ ($\therefore$ by equation (2))

$<M.\varepsilon/M$

$= \varepsilon$

(i.e.), $|S_n(x)-S_m(x)| < \varepsilon$

$\Rightarrow\{S_n\}$ converges uniformly on T

$\therefore$ $\sum f_n(x)\, g_n(x)$ converges uniformly on T

Example 5.24:

Let $F_n(x) = \sum_{k=1}^{n} e^{ikx}$

$|F_n(x)|=|\sum_{k=1}^{n} e^{ikx}| \leq 1/|\sin(x/2)|$

(i.e.) $|F_n(x)|\leq 1/|\sin(x/2)|\ \forall\ x \neq 2m\pi$, $m \Rightarrow$ integer

If $0<\delta<\pi$, we get,

$|F_n(x)| \leq 1/\sin(\delta/2)$ if $\delta \leq x \leq 2\pi-\delta$

$\therefore \{F_n\}$ is uniformly bounded on $[\delta, 2\pi-\delta]$

Let $g_n(x) = 1/n$

$\Rightarrow \{g_n\} \to 0$ uniformly on $[\delta, 2\pi-\delta]$ if $0<\delta<\pi$

By Theorem 5.22, we get,

$\sum_{n=1}^{\infty} e^{inx}/n$ converges uniformly on $[\delta, 2\pi-\delta]$ If $0<\delta<\pi$

Note:

Weierstrass M-Test cannot be used to establish the uniform convergence in the above example, Since $|e^{inx}|=1$.

Mean Convergence:

Definition 5.25:

Let $\{f_n\}$ be a sequence of Riemann-integrable functions defined on [a, b]. Assume that $f \in R$ on [a, b]. The sequence $\{f_n\}$ is said to converge in the mean to f on [a, b], and we write $\lim_{n\to\infty} f_n = f$ on [a, b] ,if $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0$

Note:

Uniform convergence of $\{f_n\}$ to f on [a, b] $\Rightarrow$ mean convergence

($\because |f(x)- f_n(x)| < \varepsilon$ for all $x \in [a, b] \Rightarrow \int_a^b |f(x)- f_n(x)|^2 dx \leq \varepsilon^2 (b-a)$

provided that each f_n is Riemann-Integrable on [a, b]

Mean Convergence $\not\Rightarrow$ Point wise convergence at any point of the interval.

For example,

For each integer $n \geq 0$, subdivide [0,1] into 2^n equal sub interval.

Let T_{2^n+k} denote that sub interval whose right end point is $\frac{(k+1)}{2^n}$

where K = 0, 1, 2, ….. 2^n-1

This yields a collection $\{I_1, I_2, \ldots\}$ of sub intervals of [0,1], of which the first few are $I_1 = [0,1]$, $I_2 = [0,1/2]$; $I_3 = [1/2,1]$; $I_4=[0,1/4]$; $I_5 = [1/4,1/2]$;

$I_6 = [1/2,3/4]$ ……

Define fn on [0,1] as follows:

$$f_n(x) = \begin{cases} 1 & if\ x \in I_n \\ 0 & if\ x \in [0,1] - I_n \end{cases}$$

Since $\int_0^1 |f_n(x)|^2\,dx$ is the length of I_n & this approaches 'o' as $n \to \infty$,

$\{f_n\}$ converges in the mean to '0'

But for each $x \in [0,1]$ we have

$\lim_{n\to\infty} \sup f_n(x) = 1$ & $\lim_{n\to\infty} \inf f_n(x) = 0$

$\therefore \{f_n(x)$ does not converge for any x in [0, 1]

Theorem 5.26:

Assume that $\underset{n\to\infty}{\text{l.i.m}}\ f_n = f$ on [a, b]. If $g \in R$ on [a, b], define $h(x)=\int_a^x f(t)g(t)dt$, $h_n(x) =\int_a^x f(t)g(t)dt$ if $x \in [a, b]$. Then $h_n \to h$ uniformly on [a, b].

Proof:

Assume $\underset{n\to\infty}{\text{l.i.m}}\ f_n = f$ on [a, b]

Let $g \in R$ on [a, b]

Define $h(x)= \int_a^x f(t)g(t)dt$ & $h_n(x) =\int_a^x f(t)g(t)$ if $x \in [a, b]$ ………(1)

To prove: $h_n \to h$ uniformly on [a, b]

Now, By Cauchy-Schwartz inequality for integrals, we get

$0 \le (\int_a^x | f(x)- f_n(x)|^2|g(t)|dt)^2 \le (\int_a^x | f(x)- f_n(x)|^2 dt).(\int_a^x |g(t)|^2 dt)$ ………(2)

Now, given $\{f_n\}$ converges in the mean to f

$\Rightarrow$ given ε> 0 there exist N such that

N>N $\Rightarrow \int_a^b |f_n(t) - f(t)|^2 dt < \varepsilon^2/A$ ....(3)

Where A $= \int_a^x |g(t)|^2 dt$

$|h(t)-h_n(t)| = |\int_a^x f(t)g(t)dt - \int_a^x f_n(t)g(t)dt|$

$= |\int_a^x [f(t)-f_n(t)]g(t)dt|$

$\leq \int_a^x |f(t)-f_n(t)||g(t)dt$

$\leq (\int_a^x |f(x)- f_n(x)|^2 dt)^{1/2}.(\int_a^x |g(t)|^2 dt)^{1/2}$ ($\because$ by(2))

$\leq (\int_a^b |f(x)- f_n(x)|^2 dt)^{1/2}.(\int_a^b |g(t)|^2 dt)^{1/2}$

$< \frac{\varepsilon}{(\int_a^b |g(t)|2dt)^{1/2}}.(\int_a^x |g(t)|^2 dt)^{1/2}$

$= \varepsilon$

(i.e.), $|h(t)-h_n(t)| < \varepsilon$

$\therefore$ $h_n \to h$ uniformly on [a, b].

Theorem 5.27:

Assume that $\underset{n\to\infty}{\text{l.i.m}}\ f_n = f$ & $\underset{n\to\infty}{\text{l.i.m}}$ Gn =g on [a,b]. Define h(x)= $\int_a^x f(t)g(t)dt$,

$h_n(x) = \int_a^x f_n(t)\, g_n(t)d$, if $x \in [a,b]$. Then $h_n \to h$ uniformly on [a, b]

Proof:

Assume that $f \in R$ & $g \in R$ on [a,b]

Assume that $\underset{n\to\infty}{\text{l.i.m}}\ f_n = f$ & $\underset{n\to\infty}{\text{l.i.m}}\ g_n = g$ on [a,b]

Given ε>0 there exist N such that

n> N $\Rightarrow \int_a^b |f_n(t) - f(t)|^2 dt < \varepsilon^2/2$ & $\int_a^b |g_n(t) - g(t)|^2 dt < \varepsilon^2/4$ …….(1)

Now, $\underset{n\to\infty}{\text{l.i.m}}\ f_n = f$ & $g \in R$ on [a b]

Then by Theorem 5.26, if x∈ [a, b]

$\int_a^x$ f(t)g(t)dt →$\int_a^x$ f(t)g(t)dt uniformly on [a, b] ………(2)

Similarly $\underset{n\to\infty}{\text{l.i.m}}$ Gn = g & f∈R on [a, b]

$\int_a^x$ f(t) g_n(t)dt →$\int_a^x$ f(t)g(t)dt uniformly on [a,b] ………(3)

By Cauchy-Schwartz inequality, we get,

$0\leq (\int_a^x |f-f_n|.|g-g_n|dt)^2 \leq (\int_a^x |f-f_n|^2dt).(\int_a^x |g-Gn|^2dt)$ ……..(4)

Now, we can write

$f_n g_n - fg = [(f- f_n)(g- g_n)] + [f_n g-fg] + [fg_n-fg]$ ..……..(5)

Now,$| h_n (t)-h (t)| = |\int_a^x f_n(t)\, g_n(t)dt - \int_a^x f(t)g(t)dt|$

$=|\int_a^x [f_n(t)\, g_n(t)-f(t)g(t)]dt|$

$\leq (\int_a^x |f-f_n|.|g-g_n|dt)+(\int_a^x f_n g\, dt - \int_a^x fg\, dt)+(\int_a^x |fg_n dt - \int_a^x fg\, dt)$ (∴by 5)

$\leq (\int_a^x |f-f_n|^2dt)^{1/2}).(\int_a^x |g-g_n|^2dt)^{1/2}+(\int_a^x f_n g\, dt - \int_a^x fg\, dt)+(\int_a^x |fg_n dt - \int_a^x fg\, dt)$

(∴ by equation (4))

= ε/2+ε/2+0+0 (∴ by equation (1), (2) and (3))

=ε (i.e.), $|h(t)-h_n(t)|< \varepsilon$

∴ $h_n \to h$ uniformly on [a, b]

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